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2-COMPETITION GRAPHS*

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Abstract. If $D = (V, A)$ is a digraph, its p-competition graph for p a positive integer has vertex set V and an edge between x and y if and only if there are distinct vertices $a_1, \cdots, a_p$ in D with (x, a_i) and (y, a_i) arcs of D for each $i = 1, \cdots, p$. This notion generalizes the notion of ordinary competition graph, which has been widely studied and is the special case where $p = 1$. Results about the case where $p = 2$ are obtained. In particular, the paper addresses the question of which complete bipartite graphs are 2-competition graphs. This problem is formulated as the following combinatorial problem: Given disjoint sets A and B such that $|A \cup B| = n$, when can one find n subsets of $A \cup B$ so that every a in A and b in B are together contained in at least two of the subsets and so that the intersection of every pair of subsets contains at most one element from A and at most one element from B?

Key words. competition graphs, edge clique coverings, food webs, complete bipartite graphs, set coverings

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1. Introduction. Suppose that $D = (V, A)$ is a digraph, loops allowed. (For all undefined graph theory terminology, see [1], [9].) If p is a positive integer, the *p-competition graph* corresponding to D, $C_p(D)$, is defined to have vertex set V and to have an edge between x and y in V if and only if, for some distinct $a_1, \cdots, a_p$ in V, (x, a_1), (y, a_1), (x, a_2), (y, a_2), $\cdots$, (x, a_p), (y, a_p) are in A. This concept was introduced in [5] as a generalization of the special case where $p = 1$, which has been studied by many authors.[3] The 1-competition graphs were motivated by a problem in ecology and have applications to a variety of fields, as summarized in [8]. The p-competition graphs have a similar motivation and similar applications to other fields. The ecological motivation is as follows: The vertices of D are considered species in an ecosystem, and there is an arc from species x to species a if x preys on a. Then x and y are joined by an edge in the p-competition graph if and only if they have at least p common prey. The literature of 1-competition graphs, otherwise known as *competition graphs*, is summarized in [4], [6], and [8]. In this paper, we study the special case where $p = 2$.

It is easy to reduce the study of p-competition graphs to a combinatorial problem that itself is of interest. Suppose that G is a graph and that $F = \{S_1, \cdots, S_r\}$ is a family of subsets of the vertex set of G, repetitions allowed. We say that F is a *p-edge clique covering*, or p-ECC, if, for every set of p distinct subscripts $i_1, i_2, \cdots, i_p$, $T = S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_p}$ is either empty or induces a clique of G, and the collection of sets of the

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[3] Those who are familiar with the competition graphs literature will note that we do not assume that the digraph D is acyclic, as is often assumed in that literature.

form T covers all edges of G. Let $\theta_e^p(G)$ be the smallest r for which there is a p-ECC F. (A 1-ECC is an ordinary edge clique covering. Edge clique coverings have played a central role in the theory of competition graphs; cf. [10].)

THEOREM 1 (see [5]). *A graph G with n vertices is a p-competition graph if and only if $\theta_e^p(G) \leqq n$.*

Proof. Suppose that $G = C_p(D)$, where $D = (V, A)$, and let $V(G) = \{v_1, \cdots, v_n\}$. For each i, let $S_i = \{v_j : (v_j, v_i) \in A\}$. It is easy to verify that the family of S_i is a p-ECC. Conversely, suppose that G and a p-ECC $F = \{S_1, \cdots, S_r\}$, $r \leqq n$, are given. Now define $D = (V, A)$ on $V = V(G)$ by letting $(v_i, v_j) \in A$ if and only if $v_i \in S_j$. It is easy to verify that $G = C_p(D)$. $\square$

COROLLARY. *A graph G with n vertices is a p-competition graph if and only if G has a p-ECC consisting of n sets.*

Proof. Suppose that F is a p-ECC of $r < n$ sets. Since repetitions are allowed in F, we can add $n - r$ copies of the empty set to F to obtain a p-ECC of size n. $\square$

Kim et al. [5] obtain a number of results about p-competition graphs in general; for example, they extend the basic results about ordinary competition graphs obtained in [2], [7], and [11]. They also obtain a variety of results about 2-competition graphs. For instance, they show that all trees are 2-competition graphs, all unicyclic graphs are 2-competition graphs except the 4-cycle C_4, and all chordal graphs are 2-competition graphs. In this paper, we study the question: What complete bipartite graphs are 2-competition graphs?

A graph $G = K_{m,x}$ is a *complete bipartite graph* if the vertices are partitioned into a pair of disjoint sets A and B of m and x vertices, respectively, and there is an edge between two vertices if and only if they are in different sets. By virtue of the corollary to Theorem 1, the question of whether $K_{m,x}$ is a 2-competition graph is reduced to the combinatorial question: If $|A \cup B| = m + x = n$, are there n subsets of $A \cup B$ (not necessarily distinct) so that (i) for all $a \in A$ and $b \in B$, a and b are contained in at least two sets, and (ii) each pair of elements from A appears together in at most one set, and similarly for each pair of elements from B? In § 2 we study this question for general m and x, showing that for fixed m, there are real numbers $a(m) < b(m) < c(m)$ so that $K_{m,x}$ is not a 2-competition graph for $x \in [a(m), b(m)]$ and $K_{m,x}$ is a 2-competition graph for $x \geqq c(m)$. In § 3 we answer the question entirely for the special case where $m = 2$. In §§ 4 and 5 we consider the special cases where $m = 3$ and $m = x$. Finally, § 6 gives closing remarks and open questions.

2. Fixed *m* and arbitrary *x*.

In this section, we study $K_{m,x}$ for arbitrary m and x. We show that for fixed m, $K_{m,x}$ is a 2-competition graph for all x sufficiently large. However, when m is sufficiently large (at least 24), we show that there is an interval of intermediate values of x for which $K_{m,x}$ is not a 2-competition graph. We do not yet have evidence to dispute the conjecture that, for all $x \geqq m \geqq 2$, if $K_{m,x}$ is a 2-competition graph, then so is $K_{m,x+1}$. Our results in the next section do prove this conjecture for $m = 2$ (though we do not have a direct proof).

THEOREM 2. *For every $m \geqq 1$, $K_{m,x}$ is a 2-competition graph for all x sufficiently large.*

Proof. Given m and x, let y be an integer such that $x \leqq y^{2m}$, let C be the set of all $(2m)$-tuples $c = (c_1, \cdots, c_{2m})$ with entries from $\{1, 2, \cdots, y\}$, and let C' be the set of all $(2m-1)$-tuples $d = (d_1, \cdots, d_{2m-1})$ with entries from $\{1, 2, \cdots, y\}$. Let B be any subset of C with $|B| = x$. Note that $|C| = y^{2m}$ and $|C'| = y^{2m-1}$. Let $G = K_{m,x}$ have one independent set $\{u_1, \cdots, u_m\}$ and the other independent set B. Build a 2-ECC for G as follows. Given $1 \leqq i \leqq 2m$ and c in B, define $c/i = (c_1, \cdots, c_{i-1}, c_{i+1}, \cdots, c_{2m})$.

Note that c/i is in C'. Given $1 \leqq i \leqq 2m$ and given d in C', let

$$S_d^i = \{u_{\lceil i/2 \rceil}\} \cup \{c \in B: c/i = d\}.$$

Thus the family of all such sets has $2my^{2m-1}$ members. To see why they form a 2-ECC, first fix u_i. Then any c in B appears with u_i in $S_{c/(2i-1)}^{2i-1}$ and $S_{c/2i}^{2i}$. Also, u_i and u_j never appear together if $i \neq j$. Finally, consider $c \neq c'$ in B. Then there is some i, so that $c_i \neq c_i'$. It follows that when $j \neq i$, then, for any d in C', either c or c' is not in S_d^j. We next show that c and c' appear together in at most one S_d^i.

Case 1. For some $j \neq i$, $c_j \neq c_j'$. Here for every d in C', either c or c' is not in S_d^i.

Case 2. For all $j \neq i$, $c_j = c_j'$. Here $c/i = c'/i$ and c and c' both appear in the set $S_{c/i}^i = S_{c'/i}^i$. However, whenever d in C' is different from $c/i = c'/i$, neither c nor c' is in S_d^i.

We conclude that G is a 2-competition graph as long as the number of sets in the family is at most the number of vertices of G; i.e.,

$$2my^{2m-1} \leqq m + x$$

or

$$2my^{2m-1} - m \leqq x. \tag{1}$$

Thus we have shown that $K_{m,x}$ is a 2-competition graph whenever

$$2my^{2m-1} - m \leqq x \leqq y^{2m}, \tag{2}$$

i.e., whenever x belongs to the interval

$$I_y = [2my^{2m-1} - m, y^{2m}].$$

Note that $I_y \neq \varnothing$ if $y \geqq 2m$. Note also that, if y is sufficiently large, say $y \geqq Y$ (where $Y \geqq 2m$), then

$$2m(y+1)^{2m-1} \leqq y^{2m},$$

and therefore

$$2m(y+1)^{2m-1} - m \leqq y^{2m}.$$

Thus, for all $y \geqq Y$, the intervals I_y and I_{y+1} overlap. It follows that, for all $x \geqq 2mY^{2m-1} - m$, $K_{m,x}$ is a 2-competition graph. □

COROLLARY. *$K_{m,y^{2m}}$ is a 2-competition graph whenever $m \geqq 1$ and $y \geqq 2m$.*

Proof. By the proof, $K_{m,x}$ is a 2-competition graph as long as (2) holds. However, (2) holds if $x = y^{2m}$ and $y \geqq 2m$. □

We now introduce the following notation, which we use throughout this section. Let $A = \{a_1, \cdots, a_m\}$ and $B = \{b_1, \cdots, b_x\}$. Let $S_1, \cdots, S_t$ be a 2-ECC for $K_{m,x}$ with bipartition A and B. Suppose that v_j is the number of sets S_i containing b_j and $v = \min v_j$. (In calculating v_j, if a set S_i and a set S_k are the same for $i \neq k$, we count them both.)

LEMMA 3. *It holds that*

$$v \geqq \frac{1 + \sqrt{1 + 8m}}{2}.$$

Proof. Given j, note that, for each i, there are two distinct subscripts $\alpha(i)$ and $\beta(i)$, so that sets $S_{\alpha(i)}$ and $S_{\beta(i)}$ both contain a_i and b_j. The pairs $\{S_{\alpha(i)}, S_{\beta(i)}\}$ are all different because, if $\{S_{\alpha(i)}, S_{\beta(i)}\} = \{S_{\alpha(k)}, S_{\beta(k)}\}$ for $i \neq k$, then a_i and a_k are in two sets together, which is impossible. It follows that

$$\binom{v_j}{2} \geqq m,$$

and so $\binom{v}{2} \geqq m$. Thus $v^2 - v - 2m \geqq 0$. Using the quadratic formula and the fact that

$$\frac{1-\sqrt{1+8m}}{2} < 0 \leqq v,$$

we conclude that $v \geqq (1 + \sqrt{1+8m})/2$. □

Remark. Suppose that u_i is the number of sets S_j containing a_i (again, with multiple counting as for v_j) and $u = \min u_i$. By symmetry,

$$u \geqq \frac{1+\sqrt{1+8x}}{2}.$$

LEMMA 4. *If $S_1, \cdots, S_t$ is a 2-ECC for $K_{m,x}$, then $t \geqq v^2x/(v + x - 1)$.*

Proof. We may assume that all of the S_i are nonempty and contain an element of B. Otherwise, we remove the empty sets and those not containing elements of B, and we still have a 2-ECC with the same v; the result follows for the original 2-ECC from the result for the new 2-ECC. Let $S'_i = S_i \cap B$. Thus we may assume that all of the S'_i are nonempty.

We next note that since each b_j is in at least v sets S'_i, we have that

$$\sum_{i=1}^{t} |S'_i| \geqq xv. \tag{3}$$

Note that no pair of elements from B is together in more than one set S'_i, and so

$$\sum_{i=1}^{t} \binom{|S'_i|}{2} \leqq \binom{x}{2}. \tag{4}$$

Using the Cauchy–Schwartz inequality and (4), we have that

$$\frac{1}{t}\left[\sum_{i=1}^{t} |S'_i|\right]^2 - \sum_{i=1}^{t} |S'_i| \leqq \sum_{i=1}^{t} |S'_i|^2 - \sum_{i=1}^{t} |S'_i| \leqq x^2 - x. \tag{5}$$

Since no S'_i is empty, $\sum_{i=1}^{t} |S'_i| \geqq t$. Then, by (5) and (3),

$$tx(x-1) \geqq \sum_{i=1}^{t} |S'_i| \left(\sum_{i=1}^{t} |S'_i| - t\right) \geqq xv(xv - t);$$

so $t \geqq v^2x/(v + x - 1)$. □

Remark. By symmetry, if u is defined as in the Remark after Lemma 3, we have that $t \geqq u^2m/(u + m - 1)$.

Remark. It follows from Lemma 4 and Theorem 1 that $K_{m,x}$ is not a 2-competition graph if

$$m + x < \frac{v^2x}{v+x-1}. \tag{6}$$

By symmetry, the same conclusion holds if

$$m+x<\frac{u^2m}{u+m-1}.$$

THEOREM 5. *For fixed $m \geqq 24$, $K_{m,x}$ is not a 2-competition graph if*

$$x\in\left[\frac{m+1}{2}-\frac{\sqrt{m^2+4m+1-2m\sqrt{1+8m}}}{2},\frac{m+1}{2}+\frac{\sqrt{m^2+4m+1-2m\sqrt{1+8m}}}{2}\right].$$

Proof. Let $\gamma = \sqrt{1+8m}$. By Lemma 3, $v \geqq (1+\gamma)/2$. Since $x \geqq 1$ (by tacit assumption) and $v \geqq 1$,

$$f_x(v) = \frac{v^2x}{v+x-1}$$

is increasing in v for fixed x. (This is easy to check by taking the first derivative.) It follows that, since $v \geqq (1+\gamma)/2$,

$$\frac{\left[\frac{1+\gamma}{2}\right]^2 x}{\frac{1+\gamma}{2}+x-1} \leqq \frac{v^2x}{v+x-1}.$$

Hence, if we can show that

$$m+x<\frac{\left[\frac{1+\gamma}{2}\right]^2 x}{\frac{1+\gamma}{2}+x-1}, \tag{7}$$

then (6) follows. However, since $x \geqq 1$ and since

$$\left[\frac{1+\gamma}{2}\right]^2 = \frac{1+\gamma}{2}+2m,$$

we see by cross-multiplying that (7) holds if and only if

$$F(x) = x^2+(-1-m)x+\left[\frac{\gamma-1}{2}\right]m<0.$$

Thus, for given m, this holds if x is between the roots of the quadratic $F(x)$, namely,

$$\frac{1+m\pm\sqrt{m^2+2m+1-2(\gamma-1)m}}{2}=\frac{m+1}{2}\pm\frac{\sqrt{m^2+4m+1-2m\sqrt{1+8m}}}{2}.$$

This proves the desired result. Note that the hypothesis $m \geqq 24$ is needed for

$$m^2+4m+1-2m\sqrt{1+8m}$$

to be nonnegative, and hence for

$$\sqrt{m^2+4m+1-2m\sqrt{1+8m}}$$

to be defined (to give a real number). For $m < 24$, the square root is undefined, and there are no real roots of the quadratic $F(x)$; hence $F(x) < 0$ is never the case. □

COROLLARY 1. *For fixed m large, $K_{m,x}$ is not a 2-competition graph for*

$$x \in \left(\frac{\sqrt{1+8m}}{2} + \frac{3}{2}, m - \frac{\sqrt{1+8m}}{2} - 1 \right).$$

Proof. Consider

$$g(m) = [m^2 + 4m + 1 - 2m\sqrt{1+8m}]^{1/2} = m[1 + 4/m + 1/m^2 - \sqrt{4/m^2 + 32/m}]^{1/2}.$$

Using the binomial theorem and the notation $o(1)$ for terms that go to zero as m goes to ∞, we find that

$$\begin{aligned} g(m) &= m\Big\{1 + \tfrac{1}{2}[4/m + 1/m^2 - \sqrt{4/m^2 + 32/m}] + \frac{\frac{1}{2}(-\frac{1}{2})}{2}[4/m + 1/m^2 \\ &\qquad - \sqrt{4/m^2 + 32/m}]^2\Big\} + o(1) \\ &= m + \tfrac{1}{2}[4 + 1/m - \sqrt{4 + 32m}] - \tfrac{1}{8}[4/\sqrt{m} + 1/m^{3/2} - \sqrt{4/m + 32}]^2 + o(1) \\ &\sim m - \sqrt{1+8m} - 2. \end{aligned}$$

Thus

$$\frac{m+1}{2} - \frac{g(m)}{2} \sim \frac{m}{2} + \frac{1}{2} - \left[\frac{m}{2} - \frac{\sqrt{1+8m}}{2} - 1\right] = \frac{\sqrt{1+8m}}{2} + \frac{3}{2},$$

and

$$\frac{m+1}{2} + \frac{g(m)}{2} \sim \frac{m}{2} + \frac{1}{2} + \left[\frac{m}{2} - \frac{\sqrt{1+8m}}{2} - 1\right] = m - \frac{\sqrt{1+8m}}{2} - \frac{1}{2}. \qquad \square$$

COROLLARY 2. *For fixed m large, $K_{m,x}$ is not a 2-competition graph for*

$$x \in (2 + m + \sqrt{1 + 2(m+1)}, \tfrac{1}{2}(m-2)^2).$$

Proof. By symmetry, Corollary 1 holds with m and x reversed. Thus, for fixed x large, $K_{m,x}$ is not a 2-competition graph for

$$m \in \left(\frac{\sqrt{1+8x}}{2} + \frac{3}{2}, x - \frac{\sqrt{1+8x}}{2} - \frac{1}{2} \right).$$

Since $\sqrt{1+8x} < \sqrt{8x} + 1$, we have that $K_{m,x}$ is not a 2-competition graph for fixed x large and

$$m \in \left(\frac{\sqrt{8x}+1}{2} + \frac{3}{2}, x - \left(\frac{\sqrt{8x}+1}{2}\right) - \frac{1}{2} \right),$$

i.e., for

$$m \in (\sqrt{2x} + 2, x - \sqrt{2x} - 1). \tag{8}$$

Note that

$$\begin{aligned} m > \sqrt{2x} + 2 &\leftrightarrow m - 2 > \sqrt{2x} \\ &\leftrightarrow x < \tfrac{1}{2}(m-2)^2. \end{aligned} \tag{9}$$

(The second equivalence follows, since $m - 2 > 0$.)

Let $z = \sqrt{x}$ (using the positive square root). Then, using the quadratic formula and the fact that $z > 0$, we have that

$$
\begin{aligned}
m < x - \sqrt{2x} - 1 &\leftrightarrow m < z^2 - \sqrt{2}z - 1 \\
&\leftrightarrow z^2 - \sqrt{2}z - (m+1) > 0 \\
&\leftrightarrow z > \tfrac{1}{2}[\sqrt{2} + \sqrt{2 + 4(m+1)}] \\
&\leftrightarrow x > 2 + m + \sqrt{1 + 2(m+1)}.
\end{aligned} \tag{10}
$$

Since $K_{m,x}$ is not a 2-competition graph for m in the interval given in (8), it follows from (9) and (10) that $K_{m,x}$ is not a 2-competition graph for

$$x \in (2 + m + \sqrt{1 + 2(m+1)}, \tfrac{1}{2}(m-2)^2). \qquad \square$$

Independently, Jacobson [3] obtained results that can be stated as follows.

THEOREM 6 (see [3]). *For fixed m large,*

(a) *$K_{m,x}$ is not a 2-competition graph if $x \in [m, (2+\sqrt{3})m)$;*

(b) *$K_{m,x}$ is a 2-competition graph if $x \in [16m^2, +\infty)$.*

Proof. (a) Jacobson proves that $K_{m,x}$ is not a 2-competition graph for sufficiently large x, $c > 2 - \sqrt{3}$, and $m = cx$, i.e., for sufficiently large m and $x < m/(2-\sqrt{3}) = (2+\sqrt{3})m$.

(b) Let $\alpha(t)$ be the smallest prime power that is at least as large as t. Since $2^r < t \leqq 2^{r+1}$ for some r, $\alpha(t) \leqq 2t$. Jacobson proves that $\theta_e^p \leqq mp(\alpha(\sqrt{x}))$ whenever $\alpha(\sqrt{x}) \geqq pm/(p-1)$. We show that if $x \geqq 16m^2$, then $2m(\alpha(\sqrt{x})) \leqq m + x$, which by Theorem 1 shows that $K_{m,x}$ is a 2-competition graph. (Note that $\alpha(\sqrt{x}) \geqq \alpha(4m) \geqq 4m > 2m$; so Jacobson's result applies.) If $x \geqq 16m^2$, we have

$$2m\alpha(\sqrt{x}) \leqq 2m(2\sqrt{x}) \leqq x < m + x. \qquad \square$$

Combining part (a) with Corollary 2 gives us that $K_{m,x}$ is not a 2-competition graph for $x \in [m, \frac{1}{2}(m-2)^2)$ when m is a fixed large number. We are not sure for what values of $x \in [\frac{1}{2}(m-2)^2, 16m^2]$ the graph $K_{m,x}$ is a 2-competition graph. Using better bounds on α (for example, those in Jacobson's paper), the constant 16 in $16m^2$ can be improved somewhat (to some value greater than or equal to 4).

3. $\mathbf{K_{2,x}}$.

In this section, we study the values of x for which $K_{2,x}$ is a 2-competition graph. Suppose that $K_{2,x}$ has one independent set $\{a, b\}$ and a second independent set $B = \{a_1, \cdots, a_x\}$, and suppose that $S_1, \cdots, S_r$ is a 2-ECC for $K_{2,x}$. Let r_a be the number of sets S_j that contain a and suppose similarly for r_b. Let s_a be the largest size of a set $B \cap S_j$ for a set S_j containing a and suppose similarly for s_b. We start with a simple lemma.

LEMMA 7. *Suppose that $S_1, \cdots, S_r$ is a 2-ECC for $K_{2,x}$. Then*

(a) $r \geqq r_a + r_b - 1$;

(b) $r_a \geqq s_a + 1$, $r_b \geqq s_b + 1$; *and*

(c) *If $s_a = 1$, then $r_a \geqq 2x$; if $s_b = 1$, then $r_b \geqq 2x$.*

Proof. (a) The elements a and b cannot be in more than one set together.

(b) Start with a set S_j containing a such that $|B \cap S_j| = s_a$. Each element of $B \cap S_j$ appears in another set S_k, $k \neq j$, containing a, and no two elements of $B \cap S_j$ can appear together in more than one set of the 2-ECC. The same is true for b.

(c) Each element of B appears twice in a set containing a and twice in a set containing b. $\square$

Continuing with the above notation, suppose that $K_{2,x}$ is a 2-competition graph and that $r = 2 + x$. Let S_i be any set containing a and b, if there is such a set (there can be at most one), and let it be any set containing a otherwise. Let $N = |B - S_i|$.

LEMMA 8. *If* $S_1, \cdots, S_{2+x}$ *is a* 2-ECC *for* $K_{2,x}$, $1 \leqq x < 15$, *and* N *is defined as above, then*

(a) $N \leqq (3+3x)/(15-x)$,

(b) $x \leqq 2N+1$, *and*

(c) $x \geqq 7$ *or* $x \leqq 3$.

Proof. (a) We can assume that each set S_j contains either a or b, since otherwise we may replace S_j by $\varnothing$ and still have a 2-ECC. Every element a_k of $B - S_i$ appears in at least two sets $S_j, j \neq i$, with a, and in at least two sets $S_j, j \neq i$, with b. If a_k is in more than two sets containing a, or more than two sets containing b, it can be deleted from one of these sets without changing the fact that we have a 2-ECC. Thus, by iterating the argument, it follows that we can assume that a_k appears in exactly two of each kind of set. Moreover, since a and b appear together at most in S_i, these four sets containing a and a_k and b and a_k have distinct subscripts. Thus every element a_k of $B - S_i$ appears in exactly four sets S_j.

Let $T_j = S_j \cap (B - S_i)$. Suppose that every $T_j, j \neq i$, is empty. Then, if $B - S_i$ is not empty, there is a vertex in B that is in none of the sets in the 2-ECC. This vertex is an isolated vertex of $K_{2,x}$, which is a contradiction. Thus $B - S_i$ must be empty, N must be 0, and so (a) follows trivially. Thus assume that some $T_j, j \neq i$, is nonempty, and therefore $B - S_i \neq \varnothing$ and $N > 0$. Without loss of generality, relabel the sets so that T_j is nonempty if and only if $j \leqq q$. Thus, since every element a_k of $B - S_i$ is in exactly four sets T_j, we have that

$$\sum_{j=1}^{q} |T_j| = 4N. \tag{11}$$

Moreover, since $T_i = \varnothing$, we have that $q < 2 + x$.

Now every pair of elements in $B - S_i$ appears in at most one T_j. Hence

$$\sum_{j=1}^{q} \binom{|T_j|}{2} \leqq \binom{N}{2}.$$

Thus

$$\sum_{j=1}^{q} |T_j|^2 - \sum_{j=1}^{q} |T_j| \leqq N(N-1).$$

Using (11), we have that

$$\sum_{j=1}^{q} |T_j|^2 \leqq N^2 + 3N. \tag{12}$$

By the Cauchy–Schwartz inequality,

$$\left[\sum_{j=1}^{q} |T_j|\right]^2 \leqq q \sum_{j=1}^{q} |T_j|^2,$$

so (since $q \geqq 1$) (12) implies that

$$\frac{1}{q}\left[\sum_{j=1}^{q} |T_j|\right]^2 \leqq N^2 + 3N.$$

Using (11), the observation that $q < 2 + x$, and the facts that $N > 0$ and $x < 15$, we have that

$$\frac{1}{q}(4N)^2 \leqq N^2 + 3N,$$

$$\frac{1}{q} \leqq \frac{1}{16} + \frac{3}{16N},$$

$$\frac{3}{16N} \geqq \frac{1}{1+x} - \frac{1}{16} = \frac{15-x}{16(1+x)},$$

$$N \leqq \frac{3+3x}{15-x}.$$

(b) Note that $|B \cap S_i| = x - N$. Now every element of $B \cap S_i$ appears in another set S_j together with a and in another set S_j together with b. Since a and b do not appear together in any sets other than S_i and since two elements of B can appear together in at most one set, it follows that all of these additional sets S_j have distinct subscripts. Hence, counting S_i, the 2-ECC has at least $2(x - N) + 1$ sets. Thus $2x - 2N + 1 \leqq 2 + x$ or $x \leqq 2N + 1$.

(c) By parts (a) and (b),

$$\frac{3+3x}{15-x} \geqq \frac{x-1}{2},$$

so $(x - 7)(x - 3) \geqq 0$. □

THEOREM 9. *$K_{2,x}$ is a 2-competition graph if and only if $x = 1$ or $x \geqq 9$.*

Proof. It is useful to consider three separate cases: (a) $x = 1$, (b) $2 \leqq x \leqq 8$, and (c) $x \geqq 9$.

(a) $\underline{x = 1.}$ Then it is trivial to show by Theorem 1 that $K_{2,x}$ is a 2-competition graph.

(b) $\underline{2 \leqq x \leqq 8.}$ Let us use the notation defined before Lemmas 7 and 8, taking $r = 2 + x$. If $s_a = 1$, then, since r_b must be at least 2, parts (a) and (c) of Lemma 7 imply that

$$2 + x = r \geqq r_a + r_b - 1 \geqq 2x + 2 - 1,$$

which is a contradiction. Thus we may assume that $s_a \geqq 2$ and, similarly, that $s_b \geqq 2$. Hence, by Lemma 7(b), $r_a \geqq 3$ and $r_b \geqq 3$.

By Lemma 8(c), $x \leqq 3$ or $x \geqq 7$. If $x = 2$, then by Lemma 7(a),

$$4 = 2 + x = r \geqq 3 + 3 - 1 = 5,$$

which is a contradiction.

Next, suppose that $x = 3$. Then, if $r_a \geqq 4$ or $r_b \geqq 4$, we have that

$$5 = 2 + x \geqq 4 + 3 - 1 = 6,$$

which is a contradiction. Thus $r_a = r_b = 3$ and, by Lemma 7(b), $s_a = s_b = 2$. Thus the sets containing a must, in their intersections with B, be the sets $\{a_1, a_2\}$, $\{a_2, a_3\}$, and $\{a_1, a_3\}$. The same is true for the sets containing b in their intersections with B. However, now two elements of B appear together in more than one set, which is a contradiction.

We now consider the case where $x = 7$. By Lemma 8(a), $N \leqq \frac{24}{8} = 3$. If $N \leqq 2$, then, by Lemma 8(b), $x \leqq 5$, which is a contradiction. Thus we have that $N = 3$. Using the notation of the proof of Lemma 8(a), we have $q < 2 + x$ sets T_j, which are subsets of a 3-element set $B - S_i$ and whose cardinalities total $4N = 12$. Moreover, any two of these sets have at most one element in common, since two elements of B appear in at most one set in common. Thus, if some T_j is all of $B - S_i$, all of the other T_j must be 1-element sets. It follows by (11) that $q \geqq 10$, which contradicts $q < 2 + x$. If all T_j are 1-element sets, then $q = 12 \geqq 2 + x$, again a contradiction. Even if some T_j's have two elements, at most three of these sets can have two elements, and then we need six more sets to get a total sum of cardinalities of 12. Hence $q \geqq 9 \geqq 2 + x$, and again there is a contradiction.

Finally, consider $x = 8$. By Lemma 8(a), $N \leqq \frac{27}{7}$, so $N \leqq 3$. By Lemma 8(b), however, $x \leqq 2N + 1 \leqq 7$, and we have a contradiction.

(c) $\underline{x \geqq 9.}$ We construct a 2-ECC E^x for $K_{2,x}$ recursively. E^x will consist of the $2 + x$ sets $R^x, K_1^x, \cdots, K_p^x, L_1^x, \cdots, L_q^x$, where

$$p = \begin{cases} x/2 + 1 & \text{if } x \text{ is even,} \\ (x+1)/2 & \text{if } x \text{ is odd;} \end{cases}$$

$$q = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

The set R^x will contain a, b, and some elements of B; the sets K_i^x will contain a and some elements of B; and the sets L_i^x will contain b and some elements of B. For $x = 9$, the sets are as follows:

$$R^9 = \{a, b, a_1, a_3, a_5, a_7, a_9\},$$

$$\begin{aligned} K_1^9 &= \{a, a_1, a_2\}, & L_1^9 &= \{b, a_2, a_3, a_6\}, \\ K_2^9 &= \{a, a_3, a_4\}, & L_2^9 &= \{b, a_2, a_4, a_7\}, \\ K_3^9 &= \{a, a_4, a_5, a_6\}, & L_3^9 &= \{b, a_1, a_4, a_8\}, \\ K_4^9 &= \{a, a_2, a_8, a_9\}, & L_4^9 &= \{b, a_5, a_8\}, \\ K_5^9 &= \{a, a_6, a_7, a_8\}, & L_5^9 &= \{b, a_6, a_9\}. \end{aligned}$$

That these sets form a 2-ECC for $K_{2,9}$ is easy to verify.

We now extend this definition recursively. If x is even, $x \geqq 10$, let

$$K_{x/2-4}^x = K_{x/2-4}^{x-1} \cup \{a_x\}, \qquad L_{x/2-1}^x = L_{x/2-1}^{x-1} \cup \{a_x\},$$

$$L_{x/2}^x = L_{x/2}^{x-1} \cup \{a_x\}, \qquad K_{x/2+1}^x = \{a, a_x\},$$

and, otherwise, let $R^x = R^{x-1}$, $K_i^x = K_i^{x-1}$, $L_i^x = L_i^{x-1}$. If x is odd, $x \geqq 11$, let

$$R^x = R^{x-1} \cup \{a_x\}, \quad K_{(x+1)/2}^x = K_{(x+1)/2}^{x-1} \cup \{a_x\}, \quad L_{(x+1)/2}^x = \{b, a_x\},$$

and, otherwise, let $K_i^x = K_i^{x-1}$, $L_i^x = L_i^{x-1}$.

Observe the following: (i) If $a_y \in R^x$, K_i^x, or L_i^x, then $y \leqq x$; (ii) If $a_y \in R^x$, then y is odd; and (iii) If $a_x \in K_i^x$ for x odd, $x \geqq 11$, then $i = (x + 1)/2$. Observation (iii) follows, since, by construction, if $i \neq (x + 1)/2$, $K_i^x = K_i^{x-1}$, and, by (i), a_x does not belong to K_i^{x-1}.

To see that we have defined a 2-ECC when $x > 9$, let us first observe that, for all $y \leqq x$, a and a_y appear in common in at least two of the sets, and b and a_y appear in common in at least two of the sets. This is because, if x is even, a and a_x appear in common in $K_{x/2-4}^x$ and $K_{x/2+1}^x$, and b and a_x appear in common in $L_{x/2-1}^x$ and

$L^x_{x/2}$; if x is odd, then a and a_x appear in common in R^x and $K^x_{(x+1)/2}$, and b and a_x appear in common in R^x and $L^x_{(x+1)/2}$. The result follows because K^z_i is a subset of K^{z+1}_i, L^z_i is a subset of L^{z+1}_i, and R^z is a subset of R^{z+1}.

Note also that a and b appear in common only in R^x. Thus it suffices to show that, if $y < z \leqq x$, then a_y and a_z appear in common in at most one set of E^x. We prove this by induction on x. It is true for $x = 9$. Assume that it is true for $x' < x$. Suppose that $y < z$. If $z < x$, then it is true for x, because, in going from E^{x-1} to E^x, neither a_y nor a_z is added to any set, and the inductive hypothesis can be applied. Thus it suffices to show this for $z = x$.

We first assume that x is odd, $x \geqq 11$. Note that a_x appears only in R^x, $K^x_{(x+1)/2}$, and $L^x_{(x+1)/2}$, and a_y is not in the last of these sets. Also, if a_y is in R^x, then, by observation (ii), y is odd. However, since x is odd and $x - 1$ is even and greater than or equal to 10,

$$K^x_{(x+1)/2} = K^{x-1}_{(x+1)/2} \cup \{a_x\} = K^{x-1}_{(x-1)/2+1} \cup \{a_x\} = \{a, a_{x-1}, a_x\}.$$

Since y is odd and $y < x$, we have that $y \neq x - 1$.

Next, suppose that x is even, $x \geqq 10$. Here a_x appears in only four sets. The case where $x = 10$ is a special case. In this case,

$$L^x_{x/2-1} = L^{x-1}_{x/2-1} \cup \{a_x\} = L^9_4 \cup \{a_{10}\} = \{b, a_5, a_8, a_{10}\},$$

$$L^x_{x/2} = L^{x-1}_{x/2} \cup \{a_x\} = L^9_5 \cup \{a_{10}\} = \{b, a_6, a_9, a_{10}\},$$

$$K^x_{x/2-4} = K^{x-1}_{x/2-4} \cup \{a_x\} = K^9_1 \cup \{a_{10}\} = \{a, a_1, a_2, a_{10}\},$$

$$K^x_{x/2+1} = \{a, a_x\} = \{a, a_{10}\}.$$

Thus, clearly, if $y < 10$, a_y appears in at most one of these sets.

The case where $x = 12$ is also a special case. In this case,

$$L^x_{x/2-1} = L^{11}_5 \cup \{a_{12}\} = L^{10}_5 \cup \{a_{12}\} = \{b, a_6, a_9, a_{10}, a_{12}\},$$

$$L^x_{x/2} = L^{11}_6 \cup \{a_{12}\} = \{b, a_{11}, a_{12}\},$$

$$K^x_{x/2-4} = K^{11}_2 \cup \{a_{12}\} = K^{10}_2 \cup \{a_{12}\} = K^9_2 \cup \{a_{12}\} = \{a, a_3, a_4, a_{12}\},$$

$$K^x_{x/2+1} = \{a, a_{12}\}.$$

Thus, if $y < 12$, a_y appears in at most one of these sets.

Finally, suppose that x is even and $x \geqq 14$. Then

$$\begin{aligned} L^x_{x/2-1} &= L^{x-1}_{x/2-1} \cup \{a_x\} \\ &= L^{x-2}_{(x-2)/2} \cup \{a_x\} \\ &= L^{x-3}_{(x-2)/2} \cup \{a_{x-2}\} \cup \{a_x\} \\ &= \{b, a_{x-3}, a_{x-2}, a_x\}, \end{aligned}$$

which holds, since $x - 3 \geqq 11$. Also,

$$\begin{aligned} L^x_{x/2} &= L^{x-1}_{x/2} \cup \{a_x\} \\ &= \{b, a_{x-1}, a_x\}, \end{aligned}$$

$$\begin{aligned} K^x_{x/2-4} &= K^{x-1}_{x/2-4} \cup \{a_x\} \\ &= K^{x-2}_{(x-2)/2-3} \cup \{a_x\} \\ &= K^{x-3}_{(x-2)/2-3} \cup \{a_x\}, \end{aligned}$$

and

$$K_{x/2+1}^{x}=\{a,a_x\}.$$

To prove that a_y and a_x are not in common in more than one of these four sets, it therefore suffices to show that $a_y \in K_{(x-2)/2-3}^{x-3}$ implies that $y < x-3$. By observation (i), however, $a_y \in K_{(x-2)/2-3}^{x-3}$ implies that $y \leqq x-3$. If $y = x-3$, then, by observation (iii), since $x-3 \geqq 11$, we must have that

$$\frac{x-2}{2}-3=\frac{(x-3)+1}{2},$$

which is false. □

4. $K_{3,x}$. In this section, we consider the case where $m = 3$. That $K_{3,1}$ is a 2-competition graph follows by a straightforward construction of a 2-ECC. It also follows from the result of [5] that every tree is a 2-competition graph. That $K_{3,2}$ is not a 2-competition graph follows from Theorem 9.

THEOREM 10. *$K_{3,3}$ is not a 2-competition graph.*

Proof. Let $K_{3,3}$ have one independent set $\{a, b, c\}$ and a second independent set $\{x, y, z\}$. If $K_{3,3}$ is a 2-competition graph, then, by the corollary to Theorem 1, there is a 2-ECC $S_1, \cdots, S_6$. We first show that each vertex of $K_{3,3}$ is contained in exactly three of the sets S_j. Now a and x are in two sets together. However, y can be in at most one of these sets, since x and y are nonadjacent. Thus a and y must be in a third set. Hence a is in at least three sets. Similarly, each vertex must be contained in at least three S_j's. Suppose that a vertex, say a, is contained in more than three S_j's, say S_1, S_2, S_3, S_4. Since b is in at least three sets, b is in one of S_1, S_2, S_3, S_4, and it cannot be in more than one of these sets, since a and b are in at most one S_j together. Thus b is in S_5 and S_6. Similarly, c is in S_5 and S_6. This, however, is impossible.

Let us suppose that a is contained in S_1, S_2, S_3 only. If b is in none of these sets, then b is in all three of S_4, S_5, S_6. However, c must either be in at least two of S_1, S_2, S_3 or in at least two of S_4, S_5, S_6. In either case, there is a contradiction, since either a and c or b and c are in two sets together. Thus we may assume that b is in one of these sets, say S_1. Then b cannot be in S_2 or S_3. Since b is in three sets, we may assume that it is also in S_4 and S_5. Similarly, c is in one of S_1, S_2, S_3 and two of S_4, S_5, S_6. Since b is in two of the latter, b and c will overlap in one of the latter, and hence c cannot be in S_1. Thus, without loss of generality, we have c in S_2, S_4, S_6. Then x must be in two sets with a, two with b, and two with c, and the only possibility is for x to be in S_1, S_2, S_4. The same argument, however, puts y in S_1, S_2, S_4. Then x and y are in two sets together, which is a contradiction. □

In the following lemma, we use the notation u defined in the remark after Lemma 3.

LEMMA 11. *If $S_1, \cdots, S_t$ is a 2-ECC for $K_{m,x}$ and $u \geqq m-1$, then*

$$t \geqq mu-\frac{m(m-1)}{2}.$$

Proof. Note that element a_1 of A must be in at least u sets of the 2-ECC; say it belongs to $S_{11}, S_{12}, \cdots, S_{1u}$. Element a_2 must be in at least u sets of the 2-ECC, and, since a_1 and a_2 can be in at most one set together, a_2 is in at least $u-1$ sets not included in the sets S_{1j}. Call these sets $S_{21}, S_{22}, \cdots, S_{2(u-1)}$. Similarly, a_3 is in at most one of the S_{1j} and at most one of the S_{2j}, so in at least $u-2$ other sets. Call these $S_{31}, S_{32}, \cdots,$

$S_{3(u-2)}$. By continuing the argument, (since $u \geqq m-1$) we find that

$$\begin{aligned} t &\geqq u+(u-1)+(u-2)+\cdots+(u-m+1) \\ &= mu-[1+2+\cdots+(m-1)] \\ &= mu-\frac{m(m-1)}{2}. \end{aligned}$$ □

THEOREM 12. *$K_{3,x}$ is not a 2-competition graph for* $x = 4, 5, 7, 8, 11$.

Proof. In addition to Lemma 11, the proof will use the following facts:

(i) $u \geqq (1+\sqrt{1+8x})/2$,

(ii) $t \geqq u^2m/(u+m-1) = f_m(u)$,

(iii) For fixed $m, f_m(u)$ is increasing in u (since $u \geqq 1$),

(iv) $m + x < t$ implies that $K_{m,x}$ is not a 2-competition graph.

Fact (i) is noted in the remark after Lemma 3, fact (ii) in the first remark before Theorem 5. Fact (iii) is easy to check by taking the derivative. Fact (iv) follows from Theorem 1.

Let $x = 4$. By (i),

$$u \geqq \frac{1+\sqrt{33}}{2} > 3,$$

and therefore $u \geqq 4$. Then, by (ii) and (iii), however,

$$t \geqq f_3(u) \geqq f_3(4) = 48/6 = 8 > m+x.$$

By (iv), $K_{3,4}$ is not a 2-competition graph.

Next, let $x = 5$. By (i),

$$u \geqq \frac{1+\sqrt{41}}{2} > 3,$$

so $u \geqq 4$. By Lemma 11, $t \geqq 12 - 3 = 9 > m + x$. By (iv), $K_{3,5}$ is not a 2-competition graph.

Suppose that $x = 7$. By (i),

$$u \geqq \frac{1+\sqrt{57}}{2} > 4,$$

so $u \geqq 5$. Then

$$t \geqq f_3(u) \geqq f_3(5) = \tfrac{75}{7},$$

so $t \geqq 11 > m + x$.

If $x = 8$, then, by (i),

$$u \geqq \frac{1+\sqrt{65}}{2} > 4,$$

so $u \geqq 5$. By Lemma 11, $t \geqq 15 - 3 = 12 > m + x$.

If $x = 11$, then, by (i),

$$u \geqq \frac{1+\sqrt{89}}{2} > 5.$$

Thus $u \geqq 6$. By Lemma 11, $t \geqq 18 - 3 = 15 > m + x$. □

Theorem 12 gives some values of x for which $K_{3,x}$ is known not to be a 2-competition graph. The next result gives values of x for which it is known to be such a graph.

THEOREM 13 (see [3]). *$K_{3,x}$ is a 2-competition graph for $x \geqq 38$.*

We do not know if $K_{3,37}$ is a 2-competition graph or if there is any $x \in (1, 38)$ such that $K_{3,x}$ is a 2-competition graph. The smallest value of x for which we do not know whether $K_{3,x}$ is a 2-competition graph is $x = 6$.

5. $K_{x,x}$. We turn now to the case where $m = x$. Theorems 9 and 10 already show that $K_{x,x}$ is not a 2-competition graph if $x = 2$ or 3. The next theorem follows by a simple argument, and therefore we include it here although it also follows from a stronger (and more difficult) result of [3], which we state below. To state the next theorem, we first need a lemma. In this lemma, we use the notation $\lceil \alpha \rceil$ to denote the least integer greater than or equal to α.

LEMMA 14. *Every positive integer x can be expressed uniquely in the form*

$$x = \binom{w-1}{2} + q, \qquad 0 < q \leqq w-1, \tag{13}$$

where

$$w = \left\lceil \frac{1+\sqrt{1+8x}}{2} \right\rceil. \tag{14}$$

Proof. Let $h(s) = (s(s-1))/2$ and let w be the smallest positive integer so that $h(w) \geqq x$. Hence, if s is such that $h(s) = x$, it follows because h is increasing for $s \geqq 1$ that $w = \lceil s \rceil$. Since

$$\binom{w-1}{2} < x \leqq \binom{w}{2},$$

it follows that x can be expressed in the form (13). By the quadratic formula, it follows that $s(s-1)/2 = x$ and $s > 0$ imply that

$$s = \frac{1+\sqrt{1+8x}}{2}.$$

This gives us (14). □

THEOREM 15. *If w and q are defined as in Lemma 14, then $K_{x,x}$ is not a 2-competition graph if $q < w/2$.*

Proof. The proof uses (i)–(iv) of the proof of Theorem 12. By (i), $u \geqq w$. By (ii) and (iii), $t \geqq f_m(u) \geqq f_m(w) = f_x(w)$. Then

$$\begin{aligned} f_x(w) > m+x = 2x &\leftrightarrow \frac{w^2 x}{w+x-1} > 2x \\ &\leftrightarrow w^2 > 2w + 2x - 2. \end{aligned}$$

Hence, by (13),

$$\begin{aligned} f_x(w) > m+x &\leftrightarrow w^2 > 2w + [(w-1)(w-2) + 2q] - 2 \\ &\leftrightarrow w > 2q \\ &\leftrightarrow q < w/2. \end{aligned}$$

The theorem follows by (iv). □

Remark. By using this theorem, we can show, for example, that $K_{x,x}$ is not a 2-competition graph for $x = 2$, 4, 7, 8, 11, 12, 16, 17, 18, 22, 23, 24, 29, 30, 31, 32, 37, 38, 39, 40. From Theorem 10, we know that this conclusion also holds for $x = 3$.

THEOREM 16 (see [3]). *$K_{x,x}$ is not a 2-competition graph for $x \geqq 4$.*

6. Closing remarks. The results in this paper leave some natural questions unresolved. For instance, the proof of Theorem 9 shows that if $K_{2,x}$ is a 2-competition graph and $x > 1$, then $K_{2,x+1}$ is a 2-competition graph. We have not been able to settle whether, for $x \geqq m > 2$, $K_{m,x}$ being a 2-competition graph implies that $K_{m,x+1}$ is a 2-competition graph. While we have determined exactly for what values of x $K_{2,x}$ is a 2-competition graph, the problem for $K_{3,x}$ remains open. In particular, small values of x such as $x = 6$ remain unresolved, as does the question of whether $K_{3,x}$ can be a 2-competition graph for any $1 < x < 38$. For the case of $K_{4,x}$, which we have not discussed in this paper, [3] shows that $K_{4,x}$ is a 2-competition graph for $x \geqq 124$ and that $K_{4,x}$ is not a 2-competition graph for $x = 4, \cdots, 10$. However, nothing else is known here.

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