Equivalence Class Universal Cycles for Permutations

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1 Introduction

In this note we describe a representation of permutations of an n-element set that can be viewed as equivalence classes of permutations of length n on n + 1 symbols. An equivalence class universal cycle is a string $x_1x_2, \ldots, x_n$ such that among the $n!$ length $n$ substrings $x_i x_{i+1}, \ldots, x_{i+n}$ (subscript addition modulo $n!$) each equivalence class is represented exactly once. We produce a complete family of $n$ such cycles. In such a family, distinct cycles use distinct representatives and each member of an equivalence class acts as representative exactly once.

The notion of universal cycles as cyclic representations of combinatorial objects, as a generalization of DeBruijn cycles, was introduced by Chung, Diaconis and Graham [1] and studied by Hurlbert [2] and Jackson [3]. The universal cycles for permutations that we examine here are one such example.

Let $\Pi^k_{i,j}$ denote the set of all $k$-permutations of $\{i, i+1, \ldots, j\}$. We write a typical element $a \in \Pi^k_{i,j}$ as a vector of $k$ distinct terms from $\{i, i+1, \ldots, j\}$. It is easy to show that universal cycles exist for $\Pi^k_{1,n}$ for $1 \leq k \leq n - 1$ and do not exist for $k = n$. (See Jackson [3].) Chung, Diaconis and Graham [1] use the concept of order isomorphism as an equivalence relation on strings from $\Pi^n_{1,m}$ to get universal cycles for $\Pi^n_{1,n}$. Such cycles exist for $m \geq 5/2n$ (Hurlbert [3]) and it is conjectured that $m = n + 1$ suffices. We consider another natural equivalence relation on $\Pi^n_{0,n}$ for which equivalence class universal cycles representing $\Pi^n_{1,n}$ exist. This gives a universal cycle for permutations of length $\{1, 2, \ldots, n\}$ using $n + 1$ symbols. Moreover, we are able to construct a complete family of such cycles.

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2 Equivalence Classes

Let \( \mathbf{1}_m = (1, 1, \ldots , 1) \) denote the vector of \( m \) ones.

**Definition 1** For \( a, b \in \Pi_{0,n}^m \)

\[
a \sim b \iff a - b \equiv k \mathbf{1}_m \pmod{n + 1} \text{ for some } k.
\]

It is easy to see that \( \sim \) is an equivalence relation and that there are \( n! \) equivalence classes corresponding to the elements of \( \Pi_{1,n}^m \). An alternative perspective on these permutations in terms of differences will prove to be useful.

**Definition 2** For \( a = (a_1, a_2, \ldots , a_m) \in \Pi_{0,n}^m \) let

\[
d(a) = ((a_2 - a_1), (a_3 - a_2), \ldots , (a_m - a_{m-1})) \in \{1, 2, \ldots , n\}^{m-1}
\]

where subtraction is modulo \( n + 1 \).

The following obvious lemma provides the connection to the equivalence relation.

**Lemma 1** \( a \sim b \iff d(a) = d(b) \)

**Lemma 2** \( a \in \Pi_{0,n}^m \) if and only if \( d(a) = (d_1, d_2, \ldots d_{m-1}) \) satisfies

\[
\sum_{k=i}^{j} d_k \not\equiv 0 \pmod{n + 1} \text{ for } 1 \leq i \leq j \leq m - 1.
\]

Proof: The \( a_i \) are distinct. \( \Box \)

In general we will say that a string \( x_1, x_2, \ldots , x_m \) of terms from \( \{1, 2, \ldots , n\} \) has property \( \mathcal{P} \) if all sums of consecutive terms (including a ‘sum’ of a single term) are distinct modulo \( n + 1 \). That is, if \( \sum_{k=i}^{j} x_k \not\equiv 0 \pmod{n + 1} \) for \( 1 \leq i \leq j \leq m - 1 \).

Denote by \( D_n \) the set of elements of \( \{1, 2, \ldots , n\}^{n-1} \) satisfying property \( \mathcal{P} \). The one to one correspondences from Lemmas 1 and 2 between permutations of \( \{1, 2, \ldots , n\} \) \( (\Pi_{1,n}^n) \), equivalence classes of \( n \)-permutations of \( \{0, 1, \ldots , n\} \) and length \( n - 1 \) vectors from \( \{1, 2, \ldots , n\} \) satisfying property \( \mathcal{P} \) \( (D_n) \) will be frequently used in what follows.
3 Difference Representations

Having set up the equivalence class partitions, with permutaions as representations, it is relatively straightforward to show the existence of universal cycles for \( D_n \) using standard techniques. We will need an additional property to ‘lift’ universal cycles for \( D_n \) to a equivalence class universal cycles for \( \Pi_{1,n}^n \).

Construct the directed graph \( G_n \) with vertices corresponding to strings in \( \{1, 2, \ldots, n\}^{n-2} \) satisfying property \( \mathcal{P} \) and arcs corresponding to elements in \( D_n \). The arc corresponding to \( d = (d_1, d_2, \ldots, d_{n-1}) \in D_n \) goes from vertex \((d_1, d_2, \ldots, d_{n-2})\) (the prefix of \( d \)) to the vertex \((d_2, d_3, \ldots, d_{n-1})\) (the suffix of \( d \)).

By \( \mathcal{P} \), the partial sums \( d_{k-1}, d_{k-2} + d_{k-1}, \ldots, d_2 + \cdots + d_{k-1}, d_1 + d_2 + \cdots + d_{k-1} \) are distinct for any \( k \). (If \( j < i \) and \( d_j + \cdots + d_{k-1} = d_i + \cdots + d_{k-1} \) then \( d_{j+1} + \cdots + d_i = 0 \) (mod \( n+1 \)).) Thus, given \( d_1, d_2, \ldots, d_k \) satisfying \( \mathcal{P} \), there are \( n-k \) choices for \( d_{k+1} \) so that \( d_1, d_2, \ldots, d_{k+1} \) satisfies \( \mathcal{P} \). In particular, there are 2 choices of \( d_{n-1} \) for any prefix. That is, the outdegree of each vertex is two. (By a symmetric argument each indegree is two.) Note also that since there are \( |D_n| = n! \) arcs, there are \( n!/2 \) vertices in \( G_n \).

Figures 1 and 2 show \( D_3 \) and \( D_4 \). The vertices are labeled by difference prefixes/suffixes and the arcs are labeled by the corresponding element of \( D_n \) and, in italics, the permutations in \( \Pi_{1,n}^n \) with this difference sequence.

The following elementary lemma shows that \( G_n \) is in general Eulerian.

**Lemma 3** \( G_n \) is strongly connected and every vertex of \( G_n \) has indegree and outdegree 2.

Proof: The statement about the degrees follows from the discussion above.

Construct the directed graph \( H_n \) with vertices corresponding to permutations in \( \Pi_{0,n}^{n-1} \) and arcs corresponding to permutations in \( \Pi_{0,n}^n \). The arc \( a = (a_1, a_2, \ldots, a_n) \in \Pi_{0,n}^n \) goes from vertex \((a_1, a_2, \ldots, a_{n-1}) \in \Pi_{0,n}^{n-1} \) to \((a_2, a_3, \ldots, a_n) \in \Pi_{0,n}^{n-1} \). It is easy to see that the indegree and outdegree of each vertex is two. It is also not difficult to show that \( H_n \) is strongly connected (see Jackson [3].) For completeness we include a short proof of this fact.

We show how to find a path between any two arcs in \( H_n \) and thus since there are no isolated vertices, \( H_n \) is strongly connected. First note that there is a path from arc \( x = (x_1, x_2, \ldots, x_n) \) to any cyclic permutation of \( x \). Namely, \((x_1, x_2, \ldots, x_n), (x_2, x_3, \ldots, x_n, x_1), \ldots, (x_i, x_{i+1}, \ldots, x_n, x_1, \ldots, x_{i-1})\). Since any permutation can be obtained from another by a sequence of transpositions of adjacent elements, it is enough to show that there is a path from \( a = (a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_n) \) to \( \hat{a} = (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n) \). Let \( b \) be the element of \( \{0, 1, \ldots, n\} \) that does not appear in \( a \). Making use of paths \( P_1, P_2, P_3 \) for cyclic permutations, we have the
path $a = (a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_n)$, $P_1$, $(a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, $(a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1}, b, a_i)$, $P_2$, $(b, a_i, a_{i+2}, \ldots, a_n, a_1, \ldots, a_{i-1}, a_{i+2}, \ldots, a_n, a_1, \ldots, a_{i-1}, a_{i+1})$, $P_3$, $(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n) = \tilde{a}$. For example, in $H_4$ we have $(1, 2, 3, 4)$, $(2, 3, 4, 1)$, $(3, 4, 1, 2)$, $(4, 1, 2, 0)$, $(1, 2, 0, 3)$, $(2, 0, 3, 1)$, $(0, 3, 1, 2)$, $(3, 1, 2, 4)$, $(1, 2, 4, 3)$.

Finally, we observe that the graph obtained from $H_n$ by identifying vertices that belong to the same equivalence class of $\Pi_{0,n}^n$ is $G_n$ and thus $G_n$ is strongly connected. Identify vertices in $H_n$ corresponding to $a, b \in \Pi_{0,n}^n$ if $d(a) = d(b)$. Each new vertex arises from an equivalence class of $n + 1$ vertices and corresponds to a string of length $n − 2$ from $\{1, 2, \ldots, n\}$ satisfying $\mathcal{P}$. That is, it corresponds to a vertex of $G_n$. Similarly, there is a correspondence between equivalence classes of arcs in $\Pi_{0,n}^n$ of $H_n$ and arcs of $G_n$. It is not difficult to check that with these correspondences $G_n$ is obtained from $H_n$. In figure 3 we give the example of $H_3$. □

**Lemma 4** Universal cycles for $D_n$ exist.

Proof: By Lemma 3, $G_n$ is Eulerian. An Eulerian cycle produces the universal cycle. □

For example, there is one Eulerian cycle in $D_3$ starting with arc 11, namely 112332. One Eulerian cycle in $D_4$ is 111242224344431213331342.

4 Universal Cycles for $\Pi_{1,n}^n$

Finally, we need to 'lift' the universal cycles for $D_n$ to equivalence class universal cycles for $\Pi_{1,n}^n$. Since we select a representative from each equivalence class, the cyclic representation must return to the same representative of each class. It is this 'lifting' that produces difficulties with the order isomorphic approach described in the introduction. For a given $a \in \{0, 1, \ldots, n\}$ and a universal cycle $u_1u_2\ldots u_n!$ for $D_n$ we construct the cycle $v_1v_2\ldots v_n!$ with $u_1 = a$ and $v_i = v_{i-1} + u_{i-1}$ for $i = 2, 3, \ldots, n!$, with addition modulo $n + 1$. For example, with the cycle 112332 for $D_3$ we get

- $a = 0 \quad 012032$
- $a = 1 \quad 123103$
- $a = 2 \quad 230210$
- $a = 3 \quad 301321$

So for $a = 0$, the equivalence class representatives are $012 \sim 123, 120 \sim 231, 203 \sim 132, 032 \sim 321, 320 \sim 213$ and $201 \sim 312$. 4
With 111242224344431213331342 for $D_4$ we get

\[
\begin{align*}
a &= 0 & 012304130421042301420143 \\
a &= 1 & 123410241032103412031204 \\
a &= 2 & 234021302143214023142310 \\
a &= 3 & 34013213204320134203421 \\
a &= 4 & 401234024310431240314032
\end{align*}
\]

Note that in both cases, every choice of $a$ ‘lifts’ to an equivalence class universal cycle. So in fact we get a family of such cycles, depending on the initial choice of $a$.

In general, $v_1 v_2 \ldots v_n!$ will be cyclic if and only if $v_1 \equiv v_n! + u_n! \pmod{n + 1}$. But, this is the same as $v_1 \equiv v_1 + u_1 + u_2 + \ldots u_n! \pmod{n + 1}$ since $v_i = v_{i-1} + u_{i-1}$. So we need the following Lemma.

**Lemma 5** Let $u_1 u_2 \ldots u_n!$ be a universal cycle for $D_n$. Then $\sum_{i=1}^{n!} u_i \equiv 0 \pmod{n+1}$.

Proof: Let $D_n^k$ denote the set of strings $d_1, \ldots, d_{n-1} \in D_n$ with $d_1 = k$. That is, those strings with lead term $k$. For $k \in \{1, 2, \ldots, n\}$, $|D_n^k| = (n - 1)!$. This follows from the counts as in the description of $G_n$ or by symmetry ($|D_n^k| = |D_n^k|$).

Let $d^{i} = (d^{i}_1, d^{i}_2, \ldots d^{i}_{n-1})$ for $i = 1, 2, \ldots n!$ be the list of the strings in $D_n$ in some order. Then since each $u_i$ is the first term of some string in $D_n$ and conversely,

\[
\sum_{i=1}^{n!} u_i = \sum_{j=1}^{n!} d^{j}_1 \\
= \sum_{k=1}^{n} \frac{|D_n^k|}{(n - 1)!} \\
= \sum_{k=1}^{n} \frac{(n + 1)n}{2} \frac{(n - 1)!}{(n - 1)!} \\
\equiv 0 \pmod{n + 1}.
\]

\[\square\]

**Theorem 1** There exists a complete family of equivalence class universal cycles for permutations of $\{1, 2, \ldots, n\}$ using the symbols $\{0, 1, 2, \ldots, n\}$.

Proof: Immediate from the above remarks and Lemmas 4 and 5. \[\square\]
Observe that by using the matrix tree theorem, counts on the number of spanning trees and hence the number of Eulerian paths in $G_n$ can be obtained. (See for example Tutte [4].) These methods would then give a count of the numbers of equivalence class universal cycles for permutations. However, it appears that it may be difficult to obtain a general expression for the evaluation of the determinant used in these counts. As is the case with de Bruijn cycles, it may be possible to use the structure of the graph $G_n$ to obtain algorithms for generating universal cycles for permutations (and hence for generating permutations).

References


