Star Avoiding Ramsey Numbers

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MCCCC Rochester October 3, 2009
Midwest Conference on Combinatorics, Computing and Cryptography

Graph Ramsey Numbers

Example

$R(C_5, K_4) = 13$

- There exists a 2-coloring of K_{12} with no red C_5 and no blue K_4.
- Every 2-coloring of K_{13} has a red C_5 or a blue K_4.

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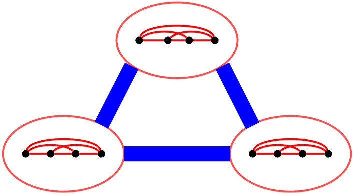

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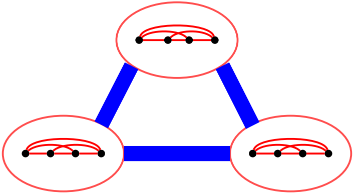

Example

'Proof' that $R(C_5, K_4) = 13$

- 2 red edges to one part $\Rightarrow$ red C_5
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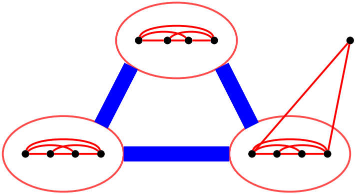

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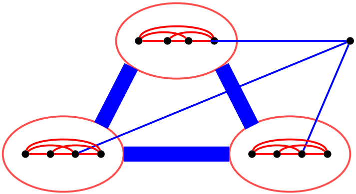
- 2 red edges to one part $\Rightarrow$ red C_5
- blue edge to each part $\Rightarrow$ blue K_4



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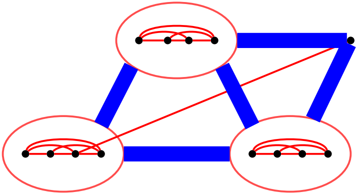

Can color 9 edges but 10th forces red C_5 or K_4

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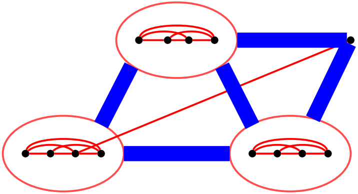

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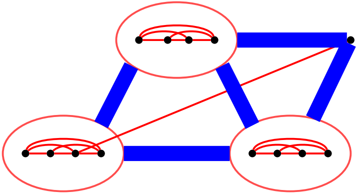

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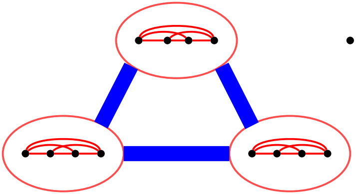

Can color 9 edges but 10th forces red C_5 or K_4
NOT a proof
Would be a proof if this is *only* good coloring of K_{12}
There are 6 critical colorings (later)

Questions

- When can we classify all sharpness examples for $R(G, H) = r$?
  - What are all good colorings of K_{r-1} (critical colorings)
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- When can we classify all sharpness examples for $R(G, H) = r$?
  - What are all good colorings of K_{r-1} (critical colorings)
- How many edges to the r^{th} vertex must be colored before a red G or blue H is forced?

A second look at our problem:

- Graph Ramsey: smallest r with no good coloring

$\ldots \quad K_{r-1}, \quad K_r, \quad K_{r+1}, \quad \ldots$

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- Size Ramsey: smallest s with no good coloring for *some* F
  $\ldots \quad |E(F)| = s - 1, \quad |E(F)| = s, \quad |E(F)| = s + 1, \quad \ldots$

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- Upper and lower Ramsey for $R(G,H) = r$:

Lower: smallest s with no good coloring for *some* F
Upper: smallest s with no good coloring for *every* F

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Restrict to $|V(F)| = r$

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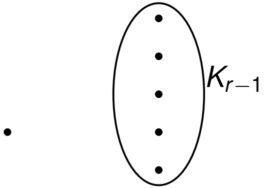
- Star avoiding Ramsey for $R(G, H) = r$:

smallest $r-1-t$ with no good coloring
$\ldots \quad K_{r-1} \setminus S(1, t-1), \quad K_{r-1} \setminus S(1, t) \quad K_{r-1}, \setminus S(1, t+1), \quad \ldots$

Star avoiding Ramsey:
$R(G, H) = r$ add/color edges to K_{r-1} one at a time:
When is a red G or blue H forced?



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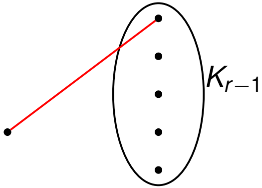

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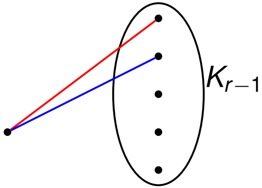

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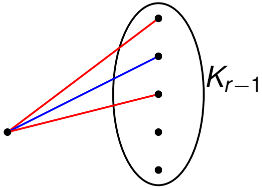

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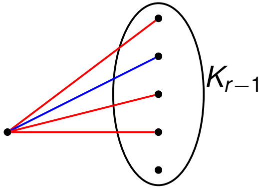

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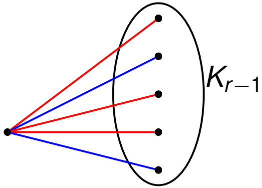

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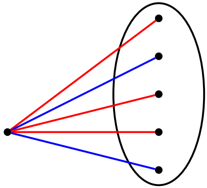

- Proofs: First classify sharpness examples
  Good colorings of K_{r-1}
- Examples with ‘few’ extra edges needed and with ‘many’ extra edges needed

Example

- $R(K_m, K_n) = r$: must add *all* $r-1$ edges (Chvatal 1974) even though we do not know what r is
-
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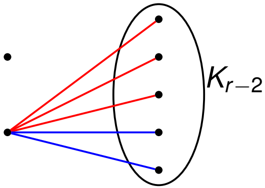

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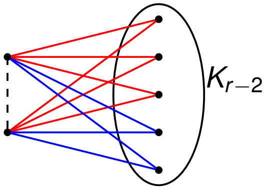

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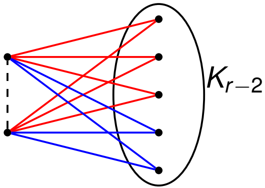
- similar for $R(mK_3, mK_3) = 5m$



Example ($R(P_n, P_3) = n$ (Gerencser and Gyrafas 1967))

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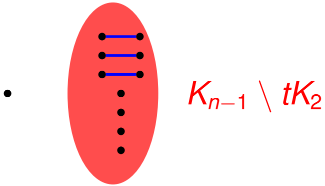

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- Sharpness examples: Blue graph is a matching plus isolated vertices
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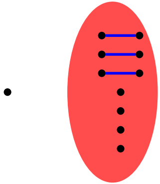

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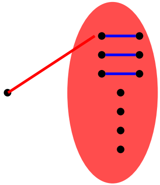

- Red edge $\Rightarrow$ red P_n
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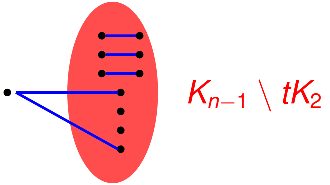
Example ($R(P_n, P_3) = n$ (Gerencser and Gyrafas 1967))

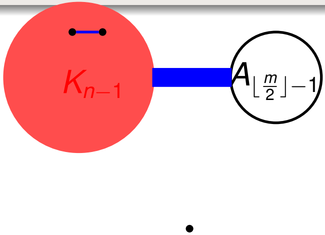
- $R(P_n, P_3) = n$
- Sharpness examples: Blue graph is a matching plus isolated vertices
- Can only add *one* edge to K_{n-1} before a red P_n or blue P_3 is forced.



- Red edge $\Rightarrow$ red P_n
- Two Blue edges $\Rightarrow$ blue P_3

Example ($R(P_n, P_m)$ (Gerencser and Gyrafas 1967))

- $R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ for $n \geq m \geq 4$
- Sharpness examples for $n \geq m + 2$. Black graph is arbitrary. Red clique can have one blue edge for odd m
- 3 other families when $n = m$ or $n = m + 1$



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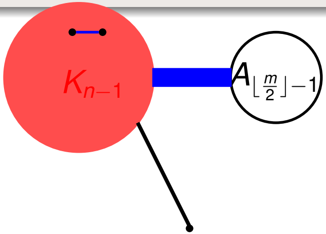

- Red or Blue edge to red K_{n-1} forces red P_n or blue P_m
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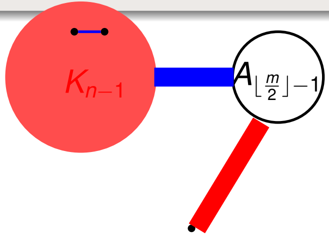

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- (only) add all red edges to $A_{\lfloor \frac{m}{2} \rfloor - 1}$

Example ($R(T_n, K_m) = (n-1)(m-2)+1$ (Chvatal 1977))

- Unique sharpness example:
  Red graph is $(m-1)K_{n-1}$ Blue graph is $K_{n-1,n-1,\ldots,n-1}$
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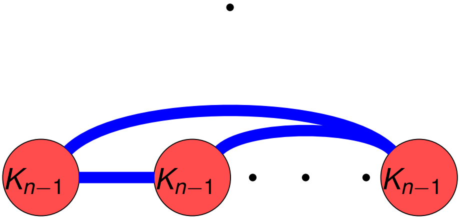

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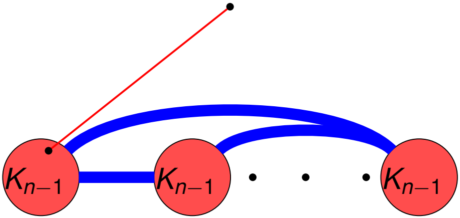

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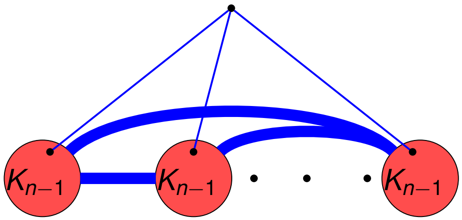

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- Blue edges to all parts $\Rightarrow$ blue K_m

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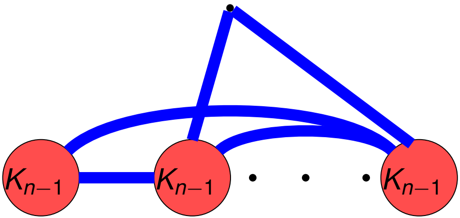
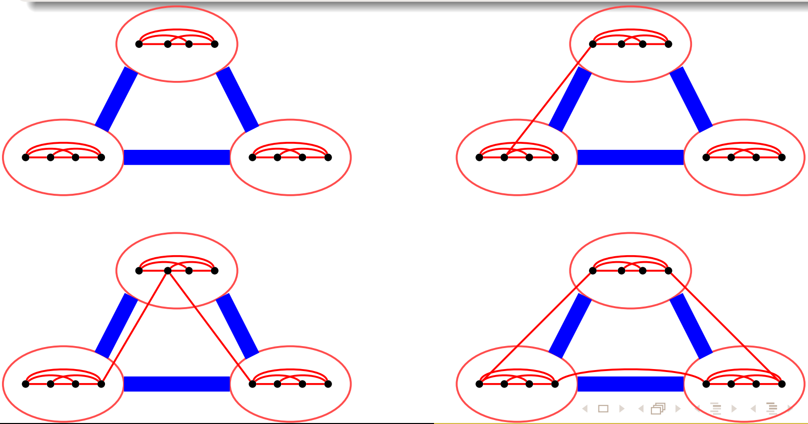
- Unique sharpness example:
  Red graph is $(m-1)K_{n-1}$ Blue graph is $K_{n-1,n-1,\ldots,n-1}$
- (only) add all $(n-1)(m-2)$ blue edges avoiding one part



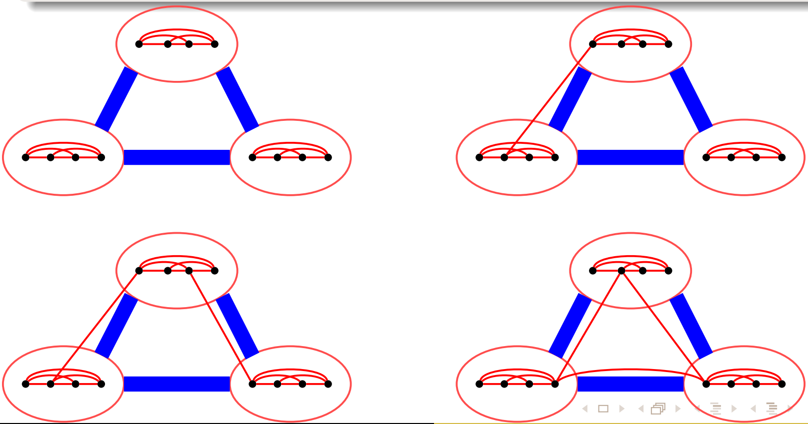
Example ($R(C_5, K_4) = 13$)

- Exactly 6 good colorings of K_{12} (Jayawardene and Rousseau 2000)
- Ends must be different (or same) for 3 extra red edges
- Extends to $R(C_n, K_4) = 3n - 2$ (but not $n = 4$)



Example ($R(C_5, K_4) = 13$)

- Exactly 6 good colorings of K_{12} (Jayawardene and Rousseau 2000)
- Ends must be the same for 3 extra red edges for $n \geq 6$
- Extends to $R(C_n, K_4) = 3n - 2$



Summary of Results

Ramsey number	Minimum Number of edges to force bad coloring
$R(mK_2, mK_2) = 3m - 1$ [L 1984]	m
$R(mK_3, mK_3) = 5m$ [BES 1975]	$5m$
$R(T_n, K_m) = (n-1)(m-1) + 1$ [C 1977]	$(n-1)(m-2) + 1$
$R(C_n, K_3) = 2n - 1$ [FS 1974]	$n + 1$
$R(C_n, K_4) = 3n - 2$ [SRM 1999]	$2n$
$R(P_n, P_3) = n$ [GG 1967]	2
$R(P_n, P_4) = n + 1$ [GG 1967]	2
$R(P_n, P_5) = n + 1$ [GG 1967]	3
$R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ [GG 1967] for all $n \geq m \geq 2$	$\lceil \frac{m}{2} \rceil$