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Bounded discrete representations of interval orders

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Abstract

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A discrete representation of an interval order $(A,\succ)$ is an interval representation for which each interval has integral endpoints. A representation is bounded if each interval is constrained with upper and lower bounds on its length. Given a finite interval order and length bounds, we give a polynomial procedure which determines whether or not it has a bounded discrete representation. The method uses Farkas' lemma to reduce the problem to finding a shortest path or detecting a negative cycle in a corresponding directed graph. Furthermore, we use this directed graph to state conditions necessary and sufficient for a representation and examine suborders which block representation in the cases with constant lower bounds of 0 or 1 and constant upper bounds.

1. Introduction

Finite interval orders are partial orders which can be represented by "strictly greater than" on a set of closed real intervals. Interval orders arise in the study of temporal events, comparison of measured properties when measurement is subject to error, in the study of preference orderings which give rise to intransitive indifference, and in the modeling of just noticeable differences in psychophysics (the study of the human perception of physical quantities such as length or sound). See the books by Fishburn [4] and Roberts [11] for discussions of these applications and reviews of related literature.

Study of interval orders as a model for temporal events began as early as Wiener

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[13] using the terminology "relations of complete sequence". (See Fishburn and Monjardet [7] for discussion of Wiener's early work on this subject.) Each temporal event corresponds to some interval in time and event a occurs before event b if a ends before b begins. This is exactly the interval order model. Such a model can be used, for instance, in chronological dating in archaeology and paleontology and in production scheduling. In each application it seems reasonable to ask that the lengths of the intervals be bounded and that the endpoints be limited to a discrete set, providing motivation for the study of bounded discrete representations.

In this paper, we examine classes of interval orders where the intervals are bounded and required to have integral endpoints.

Definition 1.1. Let $(A, \succ)$ be a finite interval order and $\boldsymbol{\alpha}, \boldsymbol{\beta}: A \to \mathbb{N}$, nonnegative integer constraints. An $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ *bounded discrete representation* of $(A, \succ)$ is a closed interval representation $J: A \to \{[l, r]: l, r \in \mathbb{Z}\}$ so that $J(i) = [l_i, r_i]$ with

(a) $i \succ j \Leftrightarrow l_i > r_j$ and

(b) $\alpha(i) \geq r_i - l_i \geq \beta(i)$ for all $i \in A$.

We will use the nonbold notation $[\alpha, \beta]$ to indicate representations for which the upper and lower bounds are constants α and β.

It is also possible to define *open* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ bounded discrete representations for which the closed intervals $J(i) = [l_i, r_i]$ are replaced with open intervals $J(i) = (l_i, r_i)$ and (a) is replaced with

(a′) $i \succ j \Leftrightarrow l_i \geq r_j$.

However, we will observe that these notions are essentially equivalent, and thus will consider only closed bounded discrete representations.

The following gives notation for interval orders which have bounded discrete representations.

Definition 1.2. Let $(A, \succ)$ be a finite interval order. $(A, \succ) \in \mathscr{D}[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ if and only if $(A, \succ)$ has an $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ bounded discrete representation.

Fishburn [3; 4, Chapter 8] makes use of Farkas' lemma to study bounded (nondiscrete) interval representations. In the nondiscrete case, by scaling, we may assume that the intervals have lengths between 1 and q. Fishburn shows that the family of minimal forbidden orders is finite if q is rational and infinite if q is irrational. (He states axioms necessary and sufficient for representation and notes the result about suborders as a comment.) A finite semiorder is an interval order with a real representation in which all the intervals have the same length. Bogart and Stellpflug [1, 2] study bounded discrete representations of semiorders and give finite lists of forbidden suborders in these cases.

Ken Bogart (personal communication) asked whether or not there is a polynomial algorithm to determine if $(A, \succ)$ is in $\mathscr{D}[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ given the order $(A, \succ)$ and the bounds $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$. In Section 3 we will give such a procedure. The procedure finds shortest paths or detects a negative cycle in a corresponding digraph $D(A, >, \alpha, \beta)$. This procedure also provides an alternative to the use of linear programming for determining if an interval order has a (nondiscrete) bounded representation that is implied in Fishburn [3]. In Section 4 we will study the digraphs $D(A, >, \alpha, \beta)$. The results from Section 4 will be used in Section 5 to state necessary and sufficient conditions for membership in $\mathscr{D}[\alpha, \beta]$ and more succinct conditions for membership in $\mathscr{D}[\alpha, 0]$ (degenerate intervals allowed) and $\mathscr{D}[\alpha, 1]$ (nondegenerate intervals) for given constants α, β.

In order to more carefully examine forbidden orders, we make the following definition for the family of minimal orders with no $[\alpha, \beta]$ representation.

Definition 1.3. Let $(A, >)$ be a finite interval order. $(A, >) \in \mathscr{F}[\alpha, \beta]$ if and only if $(A, >)$ has no $[\alpha, \beta]$ bounded discrete representation and every proper suborder $(A', >)$ of $(A, >)$ has an $[\alpha, \beta]$ bounded discrete representation. That is, $(A, >) \notin \mathscr{D}[\alpha, \beta]$ and $(A', >) \in \mathscr{D}[\alpha, \beta]$ for all $A' \subset A$.

Note that $(A, >) \notin \mathscr{D}[\alpha, \beta]$ if and only if some suborder of $(A, >)$ is isomorphic to an order in $\mathscr{F}[\alpha, \beta]$. We will show in Section 5 that $\mathscr{F}[\alpha, 0]$ is finite and that $\mathscr{F}[\alpha, 1]$ is infinite. We will also show in Section 5 that there are orders which are in both $\mathscr{F}[\alpha+1, 1]$ and $\mathscr{F}[\alpha, 1]$. Such orders have no $[\alpha+1, 1]$ bounded discrete representation and every proper suborder has an $[\alpha-1, 1]$ bounded discrete representation.

2. Preliminaries

Following Fishburn [4], we make the following formal definition of an interval order. An interval order $(A, >)$ is a set A, together with a binary relation $>$ which is irreflexive (not $a > a$ for all $a \in A$), and satisfies ($a > x$ and $b > y \Rightarrow a > y$ or $b > x$). Alternatively, there is a map J from A to a set of closed intervals denoted $J(i) = [l_i, r_i]$ in some linearly ordered set $(Y, >_0)$ such that

$$i > j \quad \Leftrightarrow \quad l_i > r_j.$$

That is, the interval for i is strictly "greater than" the interval for j. When A is countable, the linearly ordered set can be taken to be the reals under $>$. We will consider finite A and real representations. Note that condition (a) in Definition 1.1 is consistent with the definition of an interval order.

We will use the derived relations indifference ($i \sim j \Leftrightarrow$ not $i > j$ and not $j > j$), and ($i \succeq j \Leftrightarrow i > j$ or $i \sim j$). In terms of interval representations, $\sim$ and $\succeq$ satisfy

$$i \sim j \quad \Leftrightarrow \quad l_i \leq r_j \text{ and } l_j \leq r_i$$

and

$$i \succeq j \quad \Leftrightarrow \quad \text{not } j > i \quad \Leftrightarrow \quad r_i \geq l_j.$$

An (induced) suborder $(A', >')$ of an order $(A, >)$ has elements $A' \subseteq A$ and $>'$ given by the restriction of $>$ to A'.

A chain $x_1 > x_2 > \cdots > x_k$ in $(A, >)$ will be denoted $x_1 >^{k-1} x_k$. Here the superscript for $>$ indicates the number of $>$ terms appearing in the chain. Similarly, an *incomparability chain* is a sequence $x_1 \sim x_2 \sim \cdots \sim x_k$ and is denoted $x_1 \sim^{k-1} x_k$. We also use this notation for mixed chains. Thus $x >^{\pi_1} \sim^{\pi_2} >^{\pi_3} y$ would indicate a sequence of relations from x to y with the first π_1 symbols $>$, the next π_2 symbols $\sim$, and the last π_3 symbols $>$. Elements appearing in the sequence need not be distinct.

Denote the vector $(l_1, l_2, \ldots, l_n)$ by $\boldsymbol{l}$ and similarly for other variables with the length n determined by the number of such variables. (We will not distinguish between row and column vectors as this will be clear by context.) Also let $\mathbf{1}$ denote the vector with each entry 1 and similarly for other real numbers. Finally let $(\boldsymbol{l}, \boldsymbol{r})$ denote the concatenation of the vectors $\boldsymbol{l}$ and $\boldsymbol{r}$.

We will use Farkas' lemma in the following form (see e.g. Schrijver [12, p. 89]).

Lemma 2.1 (Farkas). *Exactly one of the following holds, but not both*

(a) *there exists* $\boldsymbol{x}$ *such that* $\boldsymbol{x}M \le \boldsymbol{b}$,

(b) *there exists* $\boldsymbol{c} \ge \mathbf{0}$ *such that* $M\boldsymbol{c} = \mathbf{0}$ *and* $\boldsymbol{c} \cdot \boldsymbol{b} < 0$.

We will make use of the notation and terminology for digraphs (directed graphs) found in Lawler [10].

Given a digraph D, a *circulation* is a set of nonnegative numbers (which we call *flows*) assigned to the arcs such that, for each vertex v, the sum of the flows over all arcs (w, v) "entering" v is equal to the sum of the flows over all arcs (v, w) "leaving" v. Let $c(x,y)$ be the flow on arc (x, y). Then, a circulation satisfies $\sum c(v,x) = \sum c(y, v)$ for all v, where the first sum is over all arcs (v,x) with v as the tail and the second sum is over all arcs (y, v) with v as the head. Thus, if the ordering of the columns of the vertex-arc incidence matrix M is the same as the ordering of the vector $\boldsymbol{c}$ of flows, a circulation satisfies $M\boldsymbol{c} = \mathbf{0}$.

If lengths $\boldsymbol{k}$ are assigned to the arcs of a digraph D, the total flow in a circulation $\boldsymbol{c}$ is the inner product $\boldsymbol{c} \cdot \boldsymbol{k}$. A circulation has negative total flow if this inner product is negative. If a digraph D admits a circulation with negative total flow, then it contains a negative length cycle C. We will use the notation *length*(S) to denote the length of a cycle or path S in D.

A shortest path from x to y in a digraph D, with lengths on the arcs, is a path P from x to y such that *length*(P) is less than or equal to the length of any other path from x to y. If there are no negative length cycles in D, a shortest path contains no repeated vertices. If there are negative cycles, D contains (nonsimple) paths with arbitrarily small negative length (by including many traversals of a negative cycle). Thus we consider shortest paths to be defined only if there are no negative cycles in D. We will also assume that paths contain no repeated vertices unless otherwise noted. There are many well-known polynomial algorithms which will either find the length of shortest paths between all pairs of vertices or determine that the digraph contains a negative cycle. See Lawler [10] for more details.

Fix some root v and denote the length of a shortest path from v to w by s_w.

Bellman's equations for shortest path lengths are $s_w = \min s_x + length(x, w)$, where the minimum is over all vertices x such that the arc (x, w) is in the digraph. In particular, Bellman's equations imply that for a digraph D with no negative cycles, if a vertex v is picked so that there is some path from v to every other vertex in D and if s is a vector representing shortest path lengths from v, then s is well defined and satisfies $sM \le k$. Here, as above, M is the vertex-arc incidence matrix of D and k is the vector of arc lengths.

Finally, we note the equivalence between open and closed bounded discrete interval representations.

Remark 2.2. An interval order $(A, \succ)$ has an open (α, β) discrete representation if and only if it has a closed $[\alpha-1, \beta-1]$ discrete representation. That is, there is an open interval representation if and only if there is a closed interval representation in which both upper and lower bounds are reduced by one. To see this, note that if l_i, r_i are integers for all i, $J' = \{(l_i, r_i) : i \in A\}$ satisfies the condition $i \succ j \Leftrightarrow l_i \ge r_j$ for an open interval representation if and only if $J = \{[l_i, r_i - 1] : i \in A\}$ satisfies the condition $i \succ j \Leftrightarrow l_i > r_j$ for a closed interval representation.

3. Bounded discrete representations

Clearly, $(A, \succ) \in \mathscr{D}[\alpha, \beta]$ if and only if the following integer linear programming problem, which we will call ILP, has a solution.

$\forall i \in A$	$-l_i + r_i \le \alpha(i)$:	interval length is at most $\alpha(i)$,
$\forall i \in A$	$l_i - r_i \le -\beta(i)$:	interval length is at least $\beta(i)$,
$\forall i \succ j$	$-l_i + r_j \le -1$:	$J(i)$ is greater than $J(j)$,
$\forall i \sim j$	$l_i - r_j \le 0$:	$J(i)$ is not greater than $J(j)$,
$\forall i \in A$	l_i, r_i integer.	

Note that the final inequality applied to $j \sim i$ insures also that interval $J(j)$ is not greater than $J(i)$, as is necessary for $i \sim j$. To see the third inequality, note that $i \succ j$ holds if and only if $l_i > r_j$. With the condition of integrality on the l_i and r_j, this is equivalent to $l_i \ge r_j + 1$.

Each row of the constraint matrix in ILP has exactly one -1 and one $+1$ entry. Thus, this matrix corresponds to the transpose of the vertex-arc incidence matrix of a certain directed graph.

We define the directed graph $D(A, \succ, \alpha, \beta)$ corresponding to an interval order $(A, \succ)$ and bounds $[\alpha, \beta]$ as follows. Let D have vertex set $L \cup R = \{l_1, \dots, l_{|A|}\} \cup \{r_1, \dots, r_{|A|}\}$ and arc set $U \cup V \cup W \cup Z$. The arc sets U, V, W, Z and the lengths on these arcs are

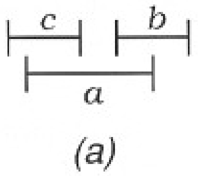
$$U = \{(l_i, r_i) : i = 1, \dots, |A|\} \quad \text{with lengths } \alpha(i),$$
$$V = \{(r_i, l_i) : i = 1, \dots, |A|\} \quad \text{with lengths } -\beta(i),$$

$$W=\{(l_i,r_j): i>j\} \qquad \text{with lengths } -1,$$
$$Z=\{(r_j,l_i): i\sim j\} \qquad \text{with lengths } 0.$$

When there is no chance of confusion, we will refer to $D(A,>,\alpha,\beta)$ as D for simplicity. For convenience, we use the same notation for variables in ILP as for the vertices of D. There is a correspondence between constraint inequalities in ILP and arcs of D, with lengths of the arcs corresponding to the right-hand side of the inequality. There are four types of inequalities and corresponding arcs; we shall refer to these as upper bounds on lengths (U), lower bounds on lengths (V), preference inequalities (W) and incomparability inequalities (Z). We will use the variables u_i, v_i, w_{ij} and z_{ij} to represent the dual variables corresponding to these inequalities. See Fig. 1 for an example of an interval representation of an interval order and its corresponding digraph.

Note that D is *bipartite*; there are no arcs joining two vertices of L or two vertices of R. An arc from L to R must be in U or W and an arc from R to L must be in V or Z.

Construct the vertex-arc incidence matrix M for the digraph described above with row j corresponding to l_j if $j\le |A|$ and corresponding to $r_{j-|A|}$ if $j>|A|$. Also order the columns so that they are partitioned with the arcs in U appearing first, the arcs



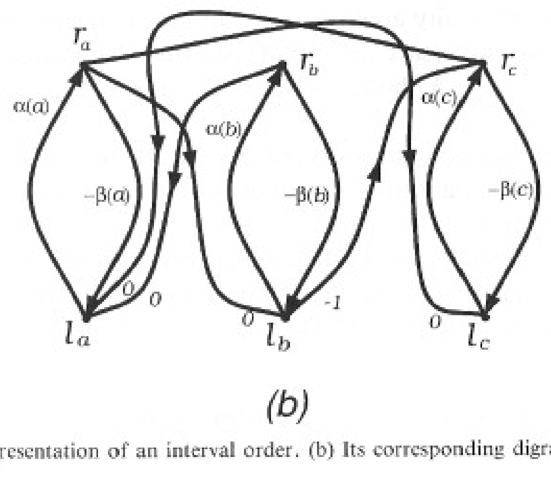


Fig. 1. (a) A representation of an interval order. (b) Its corresponding digraph $D(A,>,\alpha,\beta)$.

in V second, the arcs in W third and the arcs in Z last. Using this notation ILP becomes

$$(l,r)M \leq (\alpha, -\beta, -1, 0), \quad l_i, r_i \text{ integer}. \tag{1}$$

It is well known that vertex-arc incidence matrices are totally unimodular. Thus, since the right-hand side of (1) is integral, if there is a feasible solution to (1), then there is a feasible integral solution. So we can drop the integrality constraints. This means that regular linear programming can be used to solve the bounded discrete representation problem. However, making use of the digraph model provides a more efficient procedure to determine discrete representations and provides information on structures blocking such a representation.

Remark 3.1. We may use the ILP formulation with the cost function $\sum(r_i - l_i)$ to find a representation which minimizes the sums of the lengths. Other cost functions can also be minimized using linear programming (since total unimodularity insures integrality). However, by adding an extra element x to A such that $x \succ i$ for all remaining $i \in A$, requiring that the interval for x has length 0, and using the shortest path formulation which will be described in Corollary 3.3, we find a representation which minimizes the distance between the largest and smallest point covered by some interval without resorting to linear programming.

In order to find a more efficient procedure and to develop necessary and sufficient conditions for representability, we will apply Farkas' lemma to (1) to translate the problem of finding a bounded discrete representation for $(A, \succ)$ into the problem of finding shortest paths in D. First note that $M(u, v, w, z) = \mathbf{0}$ is the following set of equations.

$$-u_i + v_i - \sum_{j:\, i \succ j} w_{ij} + \sum_{j:\, i \sim j} z_{ji} = 0 \quad \forall i \in A, \tag{2}$$

$$u_i - v_i + \sum_{j:\, j \succ i} w_{ji} - \sum_{j:\, i \sim j} z_{ij} = 0 \quad \forall i \in A. \tag{3}$$

Note also that if we view (u, v, w, z) as the vector representing flows on arcs in U, V, W, Z, then (u, v, w, z) is a circulation and (2) represents flow conservation at vertices l_i and (3) represents flow conservation at vertices r_i. Making use of these observations we get the following.

Theorem 3.2. *Let an interval order $(A, \succ)$ and bounds $[\alpha, \beta]$ be given. $(A, \succ) \in \mathscr{D}[\alpha, \beta]$ if and only if the digraph $D(A, \succ, \alpha, \beta)$ contains no negative cycles. Furthermore, if $D(A, \succ, \alpha, \beta)$ contains no negative cycles, pick any vertex $r_v \in D$ such that v is maximal with respect to $\succ$. Then the lengths of shortest path from r_v to vertices l_i (respectively r_i) can be used as the left (respectively right) endpoints in a representation.*

Proof. $(A, \succ) \in \mathscr{D}[\alpha, \beta]$ if and only if ILP has a solution. By the definition of $D(A, \succ, \alpha, \beta)$, ILP has a solution if and only if (1) has a feasible solution. By total unimodularity and the assumption that the vector $(\boldsymbol{\alpha}, -\boldsymbol{\beta}, -\mathbf{1}, \mathbf{0})$ has integral entries, the integrality constraint on (1) can be dropped. That is, (1) without the integrality constraints has a solution if and only if it has an integral solution. By Farkas' lemma, (1) without the integrality constraints has no solution if and only if there exists a $\boldsymbol{c} = (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}) \geq \mathbf{0}$ such that $M\boldsymbol{c} = \mathbf{0}$ and $\boldsymbol{c} \cdot (\boldsymbol{\alpha}, -\boldsymbol{\beta}, -\mathbf{1}, \mathbf{0}) < 0$. Such a $\boldsymbol{c}$ represents a circulation in D, by the constraint $M\boldsymbol{c} = \mathbf{0}$. Note that $\boldsymbol{c} \cdot (\boldsymbol{\alpha}, -\boldsymbol{\beta}, -\mathbf{1}, \mathbf{0}) = \sum \alpha(i) u_i + \sum -\beta(i) v_i - \sum w_{ij}$ is the total flow of the circulation, so there exists a $\mathbf{c} \geq \mathbf{0}$ with $\boldsymbol{c} \cdot (\boldsymbol{\alpha}, -\boldsymbol{\beta}, -\mathbf{1}, \mathbf{0}) < 0$, and $M\boldsymbol{c} = \mathbf{0}$ if and only if D admits a circulation with negative total flow. If D admits a circulation with negative total flow, then it contains a negative cycle. Clearly, if D contains a negative cycle, it admits a negative circulation. So D admits a negative circulation if and only if it contains a negative cycle. This proves the first part of the theorem.

Furthermore, if D contains no negative cycles, pick some $v \in A$ which is maximal with respect to $\succ$. Recall that this means that $v \succsim i$ for all $i \in A$. Then for all $i \in A$, $(l_v, r_i) \in D$ (if $v \succ i$) or $(r_v, l_i) \in D$ (if $v \sim i$). Also, $(r_v, l_v) \in D$. Thus there is a path in D from r_v to either the l vertex or the r vertex corresponding to each element. Since $(l_i, r_i) \in D$ and $(r_i, l_i) \in D$ for all i, there is a path in D from r_v to every other vertex. Thus shortest paths from r_v to every other vertex are defined. Letting l_i (respectively r_i) be the length of a shortest path from r_v to l_i (respectively r_i) yields an integral feasible solution to (1). That this is integral follows from the integrality of the arc lengths. The inequalities (1) hold since Bellman's equations for shortest paths hold. □

Corollary 3.3. *Let an interval order $(A, \succ)$ and bounds $[\alpha, \beta]$ be given. There is a polynomial procedure to determine if $(A, \succ) \in \mathscr{D}[\alpha, \beta]$. Moreover, the procedure produces an $[\alpha, \beta]$ discrete representation if one exists.*

Proof. Construct the corresponding digraph (in polynomial time) and use any all pairs shortest path algorithm on the digraph. If a negative cycle is detected, conclude that there is no representation. Otherwise, pick any vertex x such that the shortest paths from x to every other vertex are finite, i.e., some path exists. Such a vertex exists because, as noted in the proof of the theorem, this property holds for vertices r_v corresponding to maximal elements v in the interval order. Set $J(i) = [s_{l_i}, s_{r_i}]$ where s_w denotes the length of a shortest path from vertex x to vertex w in the digraph. This is the $[\alpha, \beta]$ discrete representation. □

We note that with some modifications to ILP, this procedure works to determine representations when no integrality is expected, providing an alternative to a linear programming computation in that case.

Remark 3.4. For nonintegral closed representations, the inequalities (W) for preference become $-l_i+r_j \leq -\varepsilon$ for some small $\varepsilon>0$. This follows since a representation with $l_i>r_j$ satisfies $l_i \geq r_j+\varepsilon$ for some $\varepsilon>0$. A digraph for this nonintegral case can then be constructed putting a length of $-\varepsilon$ on the arcs from W and the algorithm in the corollary works.

By Remark 2.2, open interval representations can be transformed to closed representations in the discrete case. However, simple modifications allow direct solution in the open interval case and also allow mixes of closed, open and half open intervals. For open intervals (l_i, r_i) the condition for representation is $i \succ j \Leftrightarrow l_i \geq r_j$. Thus for discrete open intervals, the third inequality in ILP becomes $-l_i+r_j \leq 0$ for $i \succ j$ and the fourth inequality becomes $l_i - r_j \leq -1$. Similar modifications can be used if one interval is open and the other is closed.

At this point we have a polynomial algorithm to recognize if an interval order $(A, \succ) \in \mathscr{D}[\alpha, \beta]$. This answers the original question posed by Bogart. However, the digraphs $D(A, \succ, \alpha, \beta)$ provide a good deal of information. We will continue to examine bounded discrete interval orders making use of these digraphs in order to obtain necessary and sufficient conditions for membership in $\mathscr{D}[\alpha, \beta]$. More detailed descriptions of the families of minimal orders $\mathscr{F}[\alpha, 0]$ and $\mathscr{F}[\alpha, 1]$ can be found in Isaak [9].

4. Negative cycles

In this section we examine negative cycles in the digraphs $D(A, \succ, \alpha, 0)$ and $D(A, \succ, \alpha, 1)$. We will show that if there is a negative cycle in D, then there is one with certain minimal properties. We first prove a lemma about the relation between elements of A corresponding to vertices in paths in D that contain no length-α arcs corresponding to the upper bounds (U). This lemma will not require any assumption of constant bounds. In fact, the lemma does not even require the assumption of integral endpoints.

Lemma 4.1. *If $P=u, \ldots, v$ is a path in $D(A, \succ, \alpha, \beta)$ containing no arcs from U then*
(a) $u=l_i$ *and* $v=r_j \Rightarrow i \succ j$,
(b) $u=l_i$ *and* $v=l_j \Rightarrow i \succeq j$,
(c) $u=r_i$ *and* $v=l_j$ *or* $v=r_j \Rightarrow i \succeq j$.

Proof. The proof will make use of a general interval representation $\tilde{J}$ on A so that $\tilde{J}(i)=[\tilde{l}_i, \tilde{r}_i]$ and $i \succ j \Leftrightarrow \tilde{l}_i > \tilde{r}_j$. It is well known that such a real representation exists if A is finite. We first show that the left endpoints of the intervals corresponding to l vertices in P form a decreasing sequence moving along the path.

Consider any path P that begins with a vertex from L. Denote P by $l_{\sigma(1)}, r_{\sigma(2)}, l_{\sigma(3)}, r_{\sigma(4)}, \ldots, r_{\sigma(2n)}, l_{\sigma(2n+1)}$. Since there are no arcs from U, the arcs $(l_{\sigma(2k-1)}, r_{\sigma(2k)})$ must be in W, so $\sigma(2k-1) \succ \sigma(2k)$. Thus, the right endpoint of the interval for $\sigma(2k)$ is less than the left endpoint of the interval for $\sigma(2k-1)$. That is, $\tilde{r}_{\sigma(2k)} < \tilde{l}_{\sigma(2k-1)}$. Also, the arc $(r_{\sigma(2k)}, l_{\sigma(2k+1)})$ must be from V or Z. If it is from V, $\sigma(2k) = \sigma(2k+1)$. If it is from Z, $\sigma(2k) \sim \sigma(2k+1)$. In either case, $\tilde{l}_{\sigma(2k+1)} \leq \tilde{r}_{\sigma(2k)}$ and thus $\tilde{l}_{\sigma(2k+1)} < \tilde{l}_{\sigma(2k-1)}$. From this decreasing sequence, $\tilde{r}_{\sigma(1)} \geq \tilde{l}_{\sigma(1)} > \tilde{l}_{\sigma(2n+1)}$ which implies (b).

To show (a), note that (since there are no arcs from U) $(l_{\sigma(2n-1)}, r_{\sigma(2n)}) \in W$, so $\sigma(2n-1) > \sigma(2n)$ and $\tilde{r}_{\sigma(2n)} < \tilde{l}_{\sigma(2n-1)}$. As in the proof of (b), $\tilde{l}_{\sigma(2n-1)} < \tilde{l}_{\sigma(1)}$ so $\tilde{r}_{\sigma(2n)} < \tilde{l}_{\sigma(1)}$. Thus $i > j$ and (a) holds.

Finally, for (c), consider $P = r_{\sigma(0)}, l_{\sigma(1)}, \ldots, r_{\sigma(2n)}, l_{\sigma(2n+1)}$. The arc $(r_{\sigma(0)}, l_{\sigma(1)})$ is from V or Z. In either case $\sigma(0) \sim \sigma(1)$, so $\tilde{r}_{\sigma(0)} \geq \tilde{l}_{\sigma(1)}$. From the proof of (b), $\tilde{l}_{\sigma(1)} > \tilde{l}_{\sigma(2n+1)}$. So $\tilde{r}_{\sigma(0)} > \tilde{l}_{\sigma(2n+1)}$, which yields (c) when $v = l_j$. From the proof of (b), $\tilde{l}_{\sigma(1)} > \tilde{r}_{\sigma(2n)}$. So $\tilde{r}_{\sigma(0)} > \tilde{r}_{\sigma(2n)} \geq \tilde{l}_{\sigma(2n)}$, which yields (c) when $v = r_j$. □

Corollary 4.2. *Every cycle in $D(A, >, \alpha, \beta)$ must contain an arc from U.*

Proof. Assume that some cycle C contains no arc from U and reach a contradiction. Since vertices of any cycle must alternate between r vertices and l vertices, C must contain a vertex $l_{\sigma(1)}$ from L. Breaking the cycle before this vertex, we denote C by $C = l_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(2n)}, l_{\sigma(2n+1)}$ with $\sigma(2n+1) = \sigma(1)$. Then $(r_{\sigma(2n)}, l_{\sigma(1)})$ is an arc of C, so it is either from V or Z. If it is from V, then $\sigma(2n) = \sigma(1)$. If it is from Z, then $\sigma(1) \sim \sigma(2n)$. By part (a) of Lemma 4.1, $\sigma(1) > \sigma(2n)$, a contradiction in both cases. □

The corollary shows that all cycles contain at least one arc from U corresponding to the upper bound. We will show that if there is a negative cycle in $D(A, >, \alpha, 1)$, there is one such that the arcs all appear "consecutively" as a path alternating between Z arcs and U arcs. In the case that degenerate intervals are allowed, we have the following.

Lemma 4.3. *If $D(A, >, \alpha, 0)$ contains a negative cycle, then it contains a cycle C of length -1 that has exactly one arc from U.*

Proof. Let C be a negative cycle with more than one arc from U or length less than -1. We show that C can be reduced to a negative cycle C' such that C' has fewer arcs from U or C' contains exactly one arc from U and has length -1. When C already has exactly one arc from U, C' must have length -1 and one arc from U since reducing the number of arcs from U would in this case produce a negative cycle with no arcs from U, contradicting Corollary 4.2. Repeating the reduction yields the result.

Partition the cycle into paths containing exactly one U arc, with that arc appearing first in each path. Since C has negative length, one of these paths must have negative length. Pick any such negative length path $P = l_{\sigma(1)}, r_{\sigma(2)}, l_{\sigma(3)}, \ldots$ in the partition. Note that $(l_{\sigma(1)}, r_{\sigma(2)}) \in U$ and $\sigma(1) = \sigma(2)$. For $i > 1$, consider the sum of arcs

$$S(i) = \sum_{h=2}^{i} length(x_{\sigma(h-1)}, x_{\sigma(h)}) \tag{4}$$

where x may be l or r. Other than the first arc $(l_{\sigma(1)}, r_{\sigma(2)})$ with length α, the arcs are from Z or V with length 0 or from W with length -1. So $S(1) = \alpha$ and for $i > 1$, $S(i) = S(i-1)$ or $S(i) = S(i-1) - 1$. S becomes negative since P has negative length. Thus for some t, $S(t) = 0$ and $S(t+1) = -1$ with $(l_{\sigma(t)}, r_{\sigma(t+1)}) \in W$. From Lemma 4.1 (c), $\sigma(1) = \sigma(2) \geq \sigma(t+1)$. If $\sigma(1) > \sigma(t+1)$ then $(l_{\sigma(1)}, r_{\sigma(t+1)}) \in W$. In C, replace the subpath $l_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(t+1)}$ of P with $l_{\sigma(1)}, r_{\sigma(t+1)}$ to get a new cycle C' with the same length as C and one less arc from U. The lengths are the same since $S(t+1) = -1$ and $length(l_{\sigma(1)}, r_{\sigma(t+1)}) = -1$. Alternatively, *if* $\sigma(1) \sim \sigma(t+1)$ then $(r_{\sigma(t+1)}, l_{\sigma(1)}) \in Z$ and $C' = l_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(t+1)}, l_{\sigma(1)}$ is a cycle in D. The length of C' is $S(t+1) = -1$ since $length(r_{\sigma(t+1)}, l_{\sigma(1)}) = 0$. Then C' is a cycle with length -1 and exactly one arc from U. □

In order to examine the case of $[\alpha, 1]$ representations, we make the following definition for a sequence of arcs alternating between U arcs and Z arcs.

Definition 4.4. For $k \geq 1$, a path $P = l_{\sigma(1)}, r_{\sigma(2)}, \ldots, l_{\sigma(2k-1)}, r_{\sigma(2k)}$ in $D(A, \succ, \alpha, \beta)$ is a *UZ-Path* if $\sigma(2i-1) = \sigma(2i)$ for $i = 1, \ldots, k$.

As a consequence of the definition, a UZ-Path must contain arcs $(l_{\sigma(2i-1)}, r_{\sigma(2i)}) \in U$ for $i = 1, \ldots, k$. The arcs $(r_{\sigma(2i)}, l_{\sigma(2i+1)})$ for $i = 1, \ldots, k-1$ must be in Z, since otherwise, if they are in V, $\sigma(2i) = \sigma(2i+1) = \sigma(2(i+1))$, and the vertex $r_{\sigma(2i)} = r_{\sigma(2(i+1))}$ appears twice, contradicting the definition of a path. Thus $(r_{\sigma(2i)}, l_{\sigma(2i+1)}) \in Z$ for $i = 1, \ldots, k-1$ and it follows that $\sigma(2i) \sim \sigma(2i+1) = \sigma(2(i+1))$. The definition allows trivial UZ-Paths consisting of exactly one arc from U. We say that a subpath of a cycle (path) is a *maximal* UZ-Path if it is a UZ-Path and it is not included in a larger UZ-Path in the cycle (path).

In analogy to UZ-Paths, we introduce a path which alternates between arcs from W and arcs from V.

Definition 4.5. For $k \geq 1$, a path $P = l_{\sigma(1)}, r_{\sigma(2)}, \ldots, l_{\sigma(2k-1)}, r_{\sigma(2k)}$ in $D(A, \succ, \alpha, \beta)$ is a *WV-Path* if $\sigma(2i) = \sigma(2i+1)$ for $i = 1, \ldots, k-1$.

As a consequence of the definition, a WV-Path must contain arcs $(r_{\sigma(2i)}, l_{\sigma(2i+1)}) \in V$ for $i = 1, \ldots, k-1$. The arcs $(l_{\sigma(2i-1)}, r_{\sigma(2i)})$ for $i = 1, \ldots, k-1$ must be in W since otherwise, if they are in U, $\sigma(2i-1) = \sigma(2i) = \sigma(2i+1)$, and the vertex $l_{\sigma(2i+1)} = l_{\sigma(2i-1)}$ appears twice, contradicting the definition of a path. Similarly, $(l_{\sigma(2k-1)}, r_{\sigma(2k)}) \in W$, since otherwise, if it is in U, $\sigma(2(k-1)) = \sigma(2k-1) = \sigma(2k)$ and the vertex $r_{\sigma(2(k-1))} = r_{\sigma(2k)}$ appears twice, contradicting the definition of a path. So for $i = 1, \ldots, k$, $\sigma(2i-1) \succ \sigma(2i) = \sigma(2i+1)$. By transitivity, the elements in the order corresponding to a WV-Path satisfy $\sigma(1)>\sigma(3)>\cdots>\sigma(2k-1)$ and $\sigma(2k-1)>\sigma(2k)$. As with UZ-Paths, we say that a subpath is a maximal WV-Path if it is not included in a larger WV-Path.

We can now state a lemma regarding negative cycles in the case that only nondegenerate intervals are allowed.

Lemma 4.6. *If $D(A,>,\alpha,1)$ contains a negative cycle, then it contains a cycle C of length -1 that has exactly one maximal UZ-Path, or α is odd and it contains a cycle of length -2 with exactly one arc from U and exactly one maximal WV-Path.*

Proof. Let C be a negative cycle. By Corollary 4.2, C has at least one arc in U and hence at least one UZ-Path (possibly a trivial one consisting of just this arc). Let X be the property that a cycle has length -1 or -2 and has exactly one maximal UZ-Path. We first show that there is a cycle satisfying property X. Let C be a negative cycle. We shall show the following.

(a) If C contains more than one maximal UZ-Path, then C can be reduced to a negative cycle C' with the property X or with fewer maximal UZ-Paths.

(b) If C contains exactly one maximal UZ-Path, but the length of C is not -1 or -2, then C can be reduced to a negative cycle C' with property X or with exactly one maximal UZ-Path and one less arc from U.

By continuing with (a), we eventually get a cycle satisfying X or we get to a situation where we can use (b). By continuing with (b) from that point on, we eventually get a cycle with property X. This follows since, by Corollary 4.2, the reduction cannot produce a negative cycle containing no U arcs.

We prove both (a) and (b) simultaneously. Thus, start with a negative cycle C satisfying the hypothesis of (a) or (b). Partition C into paths containing exactly one maximal UZ-Path, with the maximal UZ-Path appearing first in the path. If C contains exactly one maximal UZ-Path, then the partition consists of exactly one "path" which in this case is the cycle C with the arc from Z which precedes the maximal UZ-Path deleted. Since C has negative length, one path in the partition must have negative length. Pick any such negative length path $P=l_{\sigma(1)}, r_{\sigma(2)}, l_{\sigma(3)}, \dots$ in the partition. Denote the UZ-Path at the beginning of this path by $l_{\sigma(1)}, \dots, r_{\sigma(2k)}$. Note that $\sigma(2i)=\sigma(2i-1)$ for $i=1,\dots,k$. As in equation (4) in the proof of Lemma 4.3, let $S(i)$ denote the sum of arc lengths up to the ith vertex. The arcs in the UZ-Path are from U and Z and have lengths α and 0 respectively. So $S(2k)=\alpha k$. Consider $i>2k$, that is, the part of P not containing the maximal UZ-Path. This part of the path contains no positive arcs from U since such an arc alone defines a UZ-Path. Thus, the arcs in the rest of P are from Z with length 0 and from V and W with length -1. So, for $i>2k$, $S(i)=S(i-1)$ or $S(i)=S(i-1)-1$. S becomes negative since P has negative length. Thus for some $t>2k$, $S(t)=0$ and $S(t+1)=-1$. There are two cases, depending on whether the arc causing the sum to become negative is from V or W.

Case 1: $(l_{\sigma(t)}, r_{\sigma(t+1)})\in W$. In this case, a W arc causes the sum to become negative. So $\sigma(t) > \sigma(t+1)$. There are three subcases depending on the relation between $\sigma(1)$ and $\sigma(t+1)$.

Subcase (i): $\sigma(1) > \sigma(t+1)$. In this case, replace $l_{\sigma(1)}, \ldots, r_{\sigma(t+1)}$ in C with $(l_{\sigma(1)}, r_{\sigma(t+1)}) \in W$ to get a new cycle C' with the same length as C. This follows since the arc $(l_{\sigma(1)}, r_{\sigma(t+1)})$ has length -1 and $l_{\sigma(1)}, \ldots, r_{\sigma(t+1)}$ has length $S(t+1) = -1$. C' has one less maximal UZ-Path than C.

Subcase (ii): $\sigma(1) \sim \sigma(t+1)$. In this case $(r_{\sigma(t+1)}, l_{\sigma(1)}) \in Z$ with length 0. Then $C' = l_{\sigma(1)}, \ldots, r_{\sigma(t+1)}, l_{\sigma(1)}$ is a cycle with length $S(t+1) + \mathit{length}(r_{\sigma(t+1)}, l_{\sigma(1)}) = -1 + 0 = -1$ and exactly one maximal UZ-Path. So C' has property X.

Subcase (iii): $\sigma(t+1) > \sigma(1) = \sigma(2)$. In this case, $(l_{\sigma(t+1)}, r_{\sigma(2)}) \in W$. Also, by the definition of D, $(r_{\sigma(t+1)}, l_{\sigma(t+1)}) \in V$. Let $C' = r_{\sigma(2)}, l_{\sigma(3)}, \ldots, r_{\sigma(t+1)}, l_{\sigma(t+1)}, r_{\sigma(2)}$. Note that $P' = r_{\sigma(2)}, l_{\sigma(3)}, \ldots, r_{\sigma(t+1)}$ is a subpath of P and thus is itself a path (contains no repeated vertices). To show that C' is a cycle, we must show that $l_{\sigma(t+1)}$ does not appear in P'. Note first that $l_{\sigma(t+1)}$ is not part of the maximal UZ-Path in P since, if this were the case, then $r_{\sigma(t+1)}$ would also appear on the UZ-Path (by the definition of UZ-Path), contradicting $t+1 > 2k$. If $l_{\sigma(t+1)}$ appears on the part of P' not containing the maximal UZ-Path, say as $l_{\sigma(u)}$ for $2k < u < t+1$, then by Lemma 4.1(a) applied to $l_{\sigma(u)}, \ldots, r_{\sigma(t+1)}$, $\sigma(t+1) = \sigma(u) > \sigma(t+1)$, a contradiction. Thus, $l_{\sigma(t+1)}$ does not appear on P' and C' is indeed a cycle.

C' is formed from the part of P up to $\sigma(t+1)$ with the first arc $(l_{\sigma(1)}, r_{\sigma(2)}) \in U$ deleted, shortening the path length by α. Also the two new arcs added to complete the cycle each have length -1, so the total length of C' is $S(t+1) - \alpha - 2 < 0$. C' contains exactly one maximal UZ-Path. Thus, it has fewer maximal UZ-Paths than C, unless C contained exactly one maximal UZ-Path; in the later case, C' has exactly one maximal UZ-Path and the maximal UZ-Path in C' contains one less arc from U.

Case 2: $(r_{\sigma(t)}, l_{\sigma(t+1)}) \in V$. In this case, a V arc causes the sum to become negative. So $\sigma(t) = \sigma(t+1)$. Denote by $l_{\sigma(-1)}$ and $r_{\sigma(0)}$ the two vertices preceding $l_{\sigma(1)}$ in C. That is, the two arcs preceding $(r_{\sigma(1)}, l_{\sigma(2)})$ in C are $(l_{\sigma(-1)}, r_{\sigma(0)})$ and $(r_{\sigma(0)}, l_{\sigma(1)})$. Since C is a cycle, $r_{\sigma(0)} \neq r_{\sigma(2)}$ and $\sigma(0) \neq \sigma(1)$ (since $\sigma(1) = \sigma(2)$). Thus, $(r_{\sigma(0)}, l_{\sigma(1)}) \in Z$ as it is not in V. If $(l_{\sigma(-1)}, r_{\sigma(0)}) \in U$, then $l_{\sigma(-1)}, r_{\sigma(0)}, l_{\sigma(1)}, \ldots, r_{\sigma(2k)}$ is a UZ-Path, contradicting the maximality of the UZ-Path $l_{\sigma(1)}, \ldots, r_{\sigma(2k)}$. So $(l_{\sigma(-1)}, r_{\sigma(0)}) \in W$. Then $\mathit{length}(l_{\sigma(-1)}, r_{\sigma(0)}, l_{\sigma(1)}) = 0 - 1 = -1$.

Note that $l_{\sigma(-1)}$ and $r_{\sigma(0)}$ are not equal to any of the vertices appearing on $P' = l_{\sigma(1)}, \ldots, l_{\sigma(t+1)}$. This follows immediately from the definition of a path if C contains at least two maximal UZ-Paths (since no vertices are repeated in a path and since the last vertex of P' appears before the second maximal UZ-Path).

If C contains exactly one maximal UZ-Path, P is C with one arc from Z deleted, so $\mathit{length}(P) = \mathit{length}(C)$. Now, in this case, $l_{\sigma(-1)}, r_{\sigma(0)}$ appears as $l_{\sigma(u)}, r_{\sigma(u+1)}$ in P. Thus, since $\mathit{length}(l_{\sigma(-1)}, r_{\sigma(0)}, l_{\sigma(1)}) = -1$, $\mathit{length}(C) = \mathit{length}(P) = S(u) + 1$. C satisfies the hypothesis of (b) (when C contains exactly one maximal UZ-Path), so $\mathit{length}(C) < -2$ and thus $S(u) < -3$ and $u+1 > u > t+1$. So $l_{\sigma(-1)}$ and $r_{\sigma(0)}$ are not equal to any of the vertices appearing on $P' = l_{\sigma(1)}, \ldots, l_{\sigma(t+1)}$ in the case that C contains exactly one maximal UZ-Path.

There are three possibilities for the relation between $\sigma(-1)$ and $\sigma(t)$.

Subcase (i): $\sigma(-1) \succ \sigma(t)$. In this case replace $l_{\sigma(-1)}, r_{\sigma(0)}, l_{\sigma(1)}, \ldots, r_{\sigma(t)}$ in C by $(l_{\sigma(-1)}, r_{\sigma(t)}) \in W$ to form a new cycle C' with one less maximal UZ-Path. The replaced path has length $S(t) + \mathit{length}(l_{\sigma(-1)}, r_{\sigma(0)}, l_{\sigma(1)}) = 0 - 1 = -1$. The new arc $(l_{\sigma(-1)}, r_{\sigma(t)})$ also has length -1, so the length of C' is the same as the length of C.

Subcase (ii): $\sigma(-1) \sim \sigma(t)$. In this case, $(r_{\sigma(t)}, l_{\sigma(-1)}) \in Z$. Let $C' = l_{\sigma(-1)}, r_{\sigma(0)}, l_{\sigma(1)}, \ldots, r_{\sigma(t)}, l_{\sigma(-1)}$. Note that C' has exactly one maximal UZ-Path. The length of C' is $\mathit{length}(l_{\sigma(-1)}, r_{\sigma(0)}, l_{\sigma(1)}) + S(t) + \mathit{length}(r_{\sigma(t)}, l_{\sigma(-1)}) = -1 + 0 + 0 = -1$. So C' satisfies property X.

Subcase (iii): $\sigma(t+1) = \sigma(t) \succ \sigma(-1)$. Since $(l_{\sigma(-1)}, r_{\sigma(0)}) \in W$, $\sigma(-1) \succ \sigma(0)$. By transitivity of $\succ$, $\sigma(t+1) \succ \sigma(0)$ and $(l_{\sigma(t+1)}, r_{\sigma(0)}) \in W$. This arc has length -1 and $(r_{\sigma(0)}, l_{\sigma(1)})$ has length 0. Let $C' = r_{\sigma(0)}, l_{\sigma(1)}, \ldots, r_{\sigma(t)}, l_{\sigma(t+1)}, r_{\sigma(0)}$. Then C' has exactly one maximal UZ-Path. The length of C' is $\mathit{length}(r_{\sigma(0)}, l_{\sigma(1)}) + S(t+1) + \mathit{length}(l_{\sigma(t+1)}, r_{\sigma(0)}) = 0 - 1 - 1 = -2$. So C' satisfies property X.

This completes the proof that reductions (a) and (b) can be found, and thus that there is a cycle satisfying property X.

Finally, we show that if C has exactly one maximal UZ-Path and length -2, then: when α is given, C can be reduced to a C' such that C' has exactly one maximal UZ-Path and length -1; when α is odd, C can be reduced to a C' that has exactly one maximal UZ-Path and length -1, or C' contains exactly one arc from U, exactly one maximal WV-Path and has length -2.

Let the maximal UZ-Path contain γ arcs from U and let $C = P_0, P, l_{\pi(2\gamma)}$, with the maximal UZ-Path $P_0 = l_{\pi(2\gamma)}, r_{\pi(2\gamma-1)}, \ldots, l_{\pi(2)}, r_{\pi(1)}$ and $P = l_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(v)}$ containing no arcs from U. Note that for $j = 1, \ldots, \gamma$, $(l_{\pi(2j)}, r_{\pi(2j-1)}) \in U$ and for $j = 1, \ldots, \gamma - 1$, $(r_{\pi(2j+1)}, l_{\pi(2j)}) \in Z$. Then $\mathit{length}(P_0) = \alpha\gamma$. The arcs $(r_{\pi(1)}, l_{\sigma(1)})$ and $(r_{\sigma(v)}, l_{\pi(2\gamma)})$ joining P and P_0 and P_0 to $l_{\pi(2\gamma)}$ are from Z and have length 0. Then, $\mathit{length}(C) = \mathit{length}(P_0) + \mathit{length}(P)$ and since $\mathit{length}(C) = -2$, we have $\mathit{length}(P) = -\alpha\gamma - 2$.

The path P contains no arcs from U. If P contains an arc $(r_{\sigma(w)}, l_{\sigma(w+1)}) \in Z$, then $1 < w < w+1 < v$ by the definition of P. So $P' = l_{\sigma(w-1)}, r_{\sigma(w)}, l_{\sigma(w+1)}, r_{\sigma(w+2)}$ is in P with $(l_{\sigma(w-1)}, r_{\sigma(w)}), (l_{\sigma(w+1)}, r_{\sigma(w+2)}) \in W$ (since there are no arcs from U in P). Then $\mathit{length}(P') = -1 + 0 + -1 = -2$. *By* Lemma 4.1(a) applied to P', $\sigma(w-1) \succ \sigma(w+2)$ and $(l_{\sigma(w-1)}, r_{\sigma(w+2)}) \in W$ with length -1. Replace P' in C with $l_{\sigma(w-1)}, r_{\sigma(w+2)}$ to obtain the cycle C'. The replaced path has length -2 and the new arc has length -1, so C' has length -1. Also, clearly, C' contains exactly one maximal UZ-Path P_0.

Thus, we may assume that P contains no arcs from Z. Since P also contains no arcs from U, it is a WV-Path. It is maximal since P_0 contains no arcs from W. So, for t odd $l_{\sigma(t)}$ appears in P and for t even $r_{\sigma(t)}$ appears in P. As in equation (4), let $S(i)$ denote the sum of the arcs along P, from $\sigma(1)$ to $\sigma(i)$. Since P is a WV-Path, $S(i) = -i + 1$ and since $\mathit{length}(P) = -\alpha\gamma - 2$, $v = \alpha\gamma + 3$.

Case 1: *α is even.* When α is even, note that $r_{\sigma(\alpha+2)}$ appears in P (since $\alpha + 2$ is even and since $v = \alpha\gamma + 3$). Let $P' = l_{\pi(2)}, r_{\pi(1)}, l_{\sigma(1)}, \ldots, r_{\sigma(\alpha+2)}$. Since $(l_{\pi(2)}, r_{\pi(1)}) \in U$, $\pi(2) = \pi(1)$. Then since C is a cycle, $\pi(1) \neq \sigma(1)$, and $(r_{\pi(1)}, l_{\sigma(1)}) \in Z$ with length 0. Then, $\mathit{length}(P') = \alpha + S(\alpha+2) = \alpha + -(\alpha+2) + 1 = -1$. By Lemma 4.1(c), applied to $r_{\pi(1)}, l_{\sigma(1)}, \ldots, r_{\sigma(\alpha+2)}$, $\pi(2)=\pi(1)\geq\sigma(\alpha+2)$. If $\pi(2)\succ\sigma(\alpha+2)$, then $(l_{\pi(2)}, r_{\sigma(\alpha+2)})\in W$. Replace P' in C by $l_{\pi(2)}, r_{\sigma(\alpha+2)}$ to obtain a cycle C' with the same length as C and one less arc from U. If $\pi(2)\sim\sigma(\alpha+2)$, then $(r_{\sigma(\alpha+2)}, l_{\pi(2)})\in Z$ with length 0 and $C'=P', l_{\pi(2)}$ is a cycle with exactly one arc from U (exactly one maximal UZ-Path) and $length(C')=length(P')=-1$. If C has exactly one arc from U, since C' must contain an arc from U by Corollary 4.2, the case $\pi(2)\succ\sigma(\alpha+2)$ cannot hold and it must be that $\pi(2)\sim\sigma(\alpha+2)$.

Case 2: *α is odd.* When α is odd, note that $r_{\sigma(\alpha+3)}$ appears in P (since $\alpha+3$ is even and since $v=\alpha\gamma+3$). Let $P'=l_{\pi(2)}, r_{\pi(1)}, l_{\sigma(1)}, \ldots, r_{\sigma(\alpha+3)}$. Since $(l_{\pi(2)}, r_{\pi(1)})\in U$, $\pi(2)=\pi(1)$. Then since C is a cycle, $\pi(1)\neq\sigma(1)$, and $(r_{\pi(1)}, l_{\sigma(1)})\in Z$ with length 0. Then, $length(P')=\alpha+S(\alpha+3)=\alpha+-(\alpha+3)+1=-2$. By Lemma 4.1(c), applied to $r_{\pi(1)}, l_{\sigma(1)}, \ldots, r_{\sigma(\alpha+3)}$, $\pi(2)=\pi(1)\geq\sigma(\alpha+3)$. If $\pi(2)\succ\sigma(\alpha+3)$, then $(l_{\pi(2)}, r_{\sigma(\alpha+3)})\in W$. Replace P' in C by $l_{\pi(2)}, r_{\sigma(\alpha+3)}$ to obtain a cycle C' with $length(C')=length(C)+1=-1$ and exactly one maximal UZ-Path. (The length is increased by one since $length(P')=-2$ and $length(l_{\pi(2)}, r_{\sigma(\alpha+3)})=-1$.) If $\pi(2)\sim\sigma(\alpha+3)$, then $(r_{\sigma(\alpha+3)}, l_{\pi(2)})\in Z$ with length 0 and $C'=P', l_{\pi(2)}$ is a cycle with exactly one arc from U (exactly one maximal UZ-Path), exactly one WV-Path and $length(C')=length(P')=-2$. □

We now note that negative cycles with exactly one maximal UZ-Path can be decomposed into maximal UZ-Paths and maximal WV-Paths, with the connections between these paths being arcs from Z. Consider any path P in D that contains no arcs from U. Removing all Z arcs from P produces a disconnected collection of paths alternating between arcs from W and arcs from V. Each of these subpaths (except possibly the first) must start with an arc a from W, since the arc preceding a is from Z and thus has a vertex from L as its head. So the tail of a must be in L. Since a is not in U, it must then be in W. Similarly, each of these subpaths (except possibly the last) must end in an arc a from W, since the arc following a is from Z and thus has a vertex from R as its tail. So the head of a must be in R. Since a is not in U, it must then be in W. Thus each of the subpaths except possibly the first and last is a WV-Path.

If C is a cycle containing exactly one maximal UZ-Path, the path P obtained by removing this path can be decomposed as described in the preceding paragraph. It is not difficult to see that the arc a in C following the last arc of the maximal UZ-Path and the arc b preceding the first arc of the maximal UZ-Path are from Z (since otherwise the UZ-Path would not be a path). Since a is the arc preceding the first arc of P and b is the arc following the last arc from P, in a manner similar to that in the previous paragraph, the first and last arcs in P must be in W. Thus P can be decomposed into WV-Paths and we have the following observation.

Remark 4.7. A cycle with exactly one maximal UZ-Path can be written as $C=P_0, P_1, \ldots, P_k$ where P_0 is the UZ-Path and for $i=1, \ldots, k$, P_i is a WV-Path. The last vertex of P_i is connected to the first vertex of P_{i+1} (mod $k+1$) by an arc from Z.

5. Necessary and sufficient conditions

In this section we study the negative cycles which, according to Theorem 3.2, block discrete representations. We use these cycles to obtain necessary and sufficient conditions for an order to be in $\mathscr{D}[\alpha,\beta]$. We then translate the existence of the negative cycles in $D(A,\succ,\alpha,0)$ and $D(A,\succ,\alpha,1)$ described in Lemmas 4.3 and 4.6 into a more compact set of conditions necessary and sufficient for membership in $\mathscr{D}[\alpha,0]$ and $\mathscr{D}[\alpha,1]$. We also show that $\mathscr{F}[\alpha,0]$ is finite and that $\mathscr{F}[\alpha,1]$ is infinite.

A WV-Path $l_{\sigma(1)},\dots,r_{\sigma(2k)}$ in D corresponds to a chain $\sigma(1)\succ^k\sigma(2k)$ in $(A,\succ)$. This follows since there are k arcs from W corresponding to $\succ$. *The* $k-1$ arcs from V simply correspond to $\sigma(2i-1)=\sigma(2i)$. Similarly, a UZ-Path $l_{\sigma(1)},\dots,r_{\sigma(2k)}$ corresponds to $\sigma(1)\sim^{k-1}\sigma(2k)$ since there are $k-1$ arcs from Z corresponding to $\sim$ and k arcs from U corresponding to $\sigma(2i-1)=\sigma(2i)$.

We now use the negative cycles described in Theorem 3.2 to give necessary and sufficient conditions on the order.

Theorem 5.1. $(A,\succ)\in\mathscr{D}[\alpha,\beta]$ *if and only if*

$$x\sim^{\gamma_1}\succ^{\eta_1}\sim^{\gamma_2}\succ^{\eta_2}\dots\sim^{\gamma_k}\succ^{\eta_k}y \quad\Rightarrow\quad x\succ y \tag{5}$$

holds for all integral γ_i, η_i, $k\geq 1$ *such that*

$$\sum_{i=1}^{k}(\eta_i+\beta(\eta_i-1))>\left(\sum_{i=1}^{k}\alpha(\gamma_i-1)\right)+\alpha. \tag{6}$$

Proof. By Theorem 3.2, it is enough to show that $D(A,\succ,\alpha,\beta)$ contains a negative cycle if and only if one of the conditions (5) is violated for η_i, γ_i, k satisfying (6). These conditions are simply translations of the relations implied by a negative cycle C in $D(A,\succ,\alpha,\beta)$ into chains of $\succ$ and $\sim$ in the order. In a manner similar to Remark 4.7, a negative cycle C can be decomposed as $C=P_1,P_2,\dots,P_n$, u where u is the first vertex of P_1 and the P_i are either UZ-Paths or WV-Paths. The last vertex of P_i is connected to the first vertex of P_{i+1} by an arc from Z and the last vertex of P_n is connected to the first vertex of P_1 by an arc from Z. By Corollary 4.2, C contains an arc from U and thus at least one of the paths, say P_1, is a UZ-Path. Furthermore, we may assume that no two consecutive P_i and P_{i+1}, are UZ-Paths since then P_i, P_{i+1} is itself a UZ-Path. The length of UZ-Paths is positive. Thus, if C is negative, there must be at least one WV-Path. So, we may assume that P_n is a WV-Path, since if not, we can combine P_n and P_1 into a larger UZ-Path. Let $u=l_x$ be the first vertex of P_1 and r_y be the last vertex of P_n. Then $(r_y,l_x)\in Z$ and $x\sim y$.

The sequence of paths $P_1,\dots,P_n$ and the Z arcs joining the paths translate to $xR^1\sim R^2\sim\cdots\sim R^n y$ where R^i is $\sim^{k-1}$ if P_i is a UZ-Path with k arcs from U and R^i is $\succ^k$ if P_i is a WV-Path with k arcs from W. R^1 consists of $\sim$ terms and R^n consists of $\succ$ terms by the choice of P_1 and P_n. If R^i, $i\neq 1$, consists of $\sim$ terms, combine it with the $\sim$ term preceding it and the $\sim$ term following it. This can be done since no two consecutive P_i are UZ-Paths. Then, we can write $xR^1 \sim R^2 \sim \cdots \sim R^n y$ with $\sim$ and $>$ terms alternating. Thus, if C is negative, corresponding to $P_1, P_2, \ldots, P_n$ is the chain

$$x \sim^{\gamma_1} >^{\eta_1} \sim^{\gamma_2} >^{\eta_2} \ldots \sim^{\gamma_k} >^{\eta_k} y$$

in D. Here $>^{\eta_j}$ corresponds to a WV-Path with η_j arcs from W and $\eta_j - 1$ arcs from V. So the length of this path is $-\eta_j - \beta(\eta_j - 1)$. The $\sim^{\gamma_1}$ term is $\sim^{\gamma_1 - 1} \sim$ where the $\sim^{\gamma_1 - 1}$ corresponds to the first UZ-Path with γ_1 arcs from U followed by the $\sim$ term for the first Z arc linking P_1 to P_2. This subpath has length $\alpha\gamma_1$. For $j \neq 1$, the term $\sim^{\gamma_j}$ corresponds to a Z arc linking two WV-Paths if $\gamma_j = 1$. If $\gamma_j > 1$, the term is $\sim \sim^{\gamma_j - 2} \sim$. In this case, the first and last $\sim$ correspond to the linking Z arcs and $\sim^{\gamma_j - 2}$ corresponds to a UZ-Path with $\gamma_j - 1$ arcs from U and $\gamma_j - 2$ arcs from Z. Such a subpath has length $\alpha(\gamma_j - 1)$.

Summing the lengths for the P_i noted above, we get

$$\begin{aligned} &-\sum_{i=1}^{k} (\eta_i + \beta(\eta_i - 1)) + \left(\sum_{i=2}^{k} \alpha(\gamma_i - 1) \right) + \alpha\gamma_1 \\ &\quad = -\sum_{i=1}^{k} (\eta_i + \beta(\eta_i - 1)) + \left(\sum_{i=1}^{k} \alpha(\gamma_i - 1) \right) + \alpha. \end{aligned} \tag{7}$$

If $(A, >) \notin \mathscr{D}[\alpha, \beta]$, then there is a cycle C of negative length. It follows that $P_1, \ldots, P_n$ has negative length, so

$$-\sum_{i=1}^{k} (\eta_i + \beta(\eta_i - 1)) + \left(\sum_{i=1}^{k} \alpha(\gamma_i - 1) \right) + \alpha < 0,$$

and (6) holds. However, since P_n is joined to P_1 by a Z arc, we have $y \sim x$. Thus (5) fails.

Conversely, suppose that (6) holds but (5) fails for a sequence of relations. If (5) fails with $y \sim x$, then $x \sim^{\gamma_1} >^{\eta_1} \ldots \sim^{\gamma_k} >^{\eta_k} y \sim x$ gives a closed directed path whose length is given in equation (7). By (6), this length is less than zero. A negative closed directed path contains a negative cycle, thus, $(A, >) \notin \mathscr{D}[\alpha, \beta]$. If (5) fails with $y > x$, then $(l_y, r_x) \in W$. This arc is in D and has length -1. Let P_i be the paths defined from $x \sim^{\gamma_1} >^{\eta_1} \ldots \sim^{\gamma_k} >^{\eta_k} y$ as above and $P = P_1, \ldots, P_n$ as above. Then (7) gives the length of P and by (6), this length is negative. Also, let P_1' be P_1 with the first arc (l_x, r_x), having length α, removed. So P_1' starts with vertex r_x. Then $\mathit{length}(P_1') < \mathit{length}(P_1)$. Recall that r_y is the last vertex in P_n. Then $C' = P_1', \ldots, P_n, l_y, r_x$ has negative length because $P = P_1, \ldots, P_n$ has negative length and $\mathit{length}(P_1') < \mathit{length}(P_1)$ and $\mathit{length}(l_y, r_x) = -1$. Either C' is a negative cycle or it contains a negative cycle (if the new vertex l_y also appears earlier in C). In either case, D contains a negative cycle and $(A, >) \notin \mathscr{D}[\alpha, \beta]$. $\square$

The conditions in the previous theorem are not independent. In the cases that the lower bounds are 0 or 1, we can use the structure of the negative cycles in Lemmas 4.3 and 4.6 to state more concise conditions.

Theorem 5.2. $(A, \succ) \in \mathscr{D}[\alpha, 0]$ *if and only if*

$$x \sim \succ^{\eta_1} \sim \succ^{\eta_2} \dots \sim \succ^{\eta_k} y \quad \Rightarrow \quad x \succ y \tag{8}$$

holds for all integral $\eta_i \geq 1$, *such that*

$$\sum_{i=1}^{k} \eta_i = \alpha + 1. \tag{9}$$

Proof. By Theorem 3.2, it is enough to show that $D(A, \succ, \alpha, 0)$ contains a negative cycle if and only if one of the conditions is violated. Suppose that there is a negative cycle C. We translate the relations implied by a negative cycle $C = l_x, r_x, P, l_x$ of the type described in Lemma 4.3 into chains in the order. C contains one arc (l_x, r_x) from U connected by an arc from Z to a sequence P of WV-Paths each also joined by an arc from Z. If y is the element corresponding to the last vertex r_y in P, then $x \sim \succ^{\eta_1} \sim \succ^{\eta_2} \dots \sim \succ^{\eta_k} y$ holds in the order. Here, the η_i indicate the number of arcs from W in the WV-Paths. The WV-Paths are nonempty, so $\eta_i \geq 1$ for all i. The only arcs with nonzero length in C are those from W with length -1 and the one arc from U with length α. From Lemma 4.3 the cycle has length -1 and so there are $\alpha + 1$ arcs from W. Thus, $\sum_{i=1}^{k} \eta_i = \alpha + 1$ and (9) holds. Completing the negative cycle C is an arc $(r_y, l_x) \in Z$. This corresponds to $x \sim y$, violating (8).

Conversely, assume that (9) holds but (8) fails for some sequence of relations. By the correspondence between $x \sim \succ^{\eta_1} \dots \sim \succ^{\eta_k} y$ and the path l_x, r_x, P in C (this can be shown to be a path by Lemma 4.1), if (9) holds and (8) fails with $y \succ x$, we reach a contradiction. The contradiction is reached because by Lemma 4.1(c) applied to r_x, P yields $x \succeq y$ (since r_y is the last vertex in P). Thus it must be that (8) fails with $x \sim y$. Then D contains a negative cycle l_x, r_x, P, r_y, l_x. This is a cycle since $x \sim y$ implies that $(r_y, l_x) \in Z$. $\square$

Theorem 5.3. *For*

$$x \sim^{\gamma} \succ^{\eta_1} \sim \succ^{\eta_2} \dots \sim \succ^{\eta_k} y \quad \Rightarrow \quad x \succ y \tag{10}$$

and

$$\sum_{i=1}^{k} (2\eta_i - 1) = \gamma\alpha + 1, \tag{11}$$

we have the following.

(a) *When* α *is even*: $(A, \succ) \in \mathscr{D}[\alpha, 1]$ *if and only if* (10) *holds for all integral* $\eta_i \geq 1$, $\gamma \geq 1$ *satisfying* (11).

(b) *When* α *is odd*: $(A, \succ) \in \mathscr{D}[\alpha, 1]$ *if and only if* (10) *holds for all integral* $\eta_i \geq 1$, $\gamma \geq 1$ *satisfying* (11) *and*

$$x \sim \succ^{(\alpha+3)/2} y \quad \Rightarrow \quad x \succ y \tag{12}$$

holds.

Proof. By Theorem 3.2, it is enough to show that $D(A, \succ, \alpha, 1)$ contains a negative cycle if and only if one of the conditions is violated. Suppose there is a negative cycle. We show that if α is even, (10) fails for some integral $\eta_i \geq 1$, $\gamma \geq 1$ satisfying (11), and if α is odd (10) fails for some integral $\eta_i \geq 1$, $\gamma \geq 1$ satisfying (11) or (12) fails. We translate the relations implied by a negative cycle of the type described in Lemma 4.6 into chains in the order. Such a negative cycle contains one maximal UZ-Path connected by an arc from Z to a sequence of maximal WV-Paths each also joined by an arc from Z. Denote this by $C = P_0, P, l_x$ where P_0 is the UZ-Path, P is a sequence of WV-Paths joined by arcs from Z, and l_x is the first vertex of P_0. Let y denote the element corresponding to the last vertex r_y in P. Then $x \sim^{\gamma-1} \sim \succ^{\eta_1} \sim \succ^{\eta_2} \cdots \sim \succ^{\eta_k} y$ holds in the order. Here, the η_i indicate the number of arcs from W in the WV-Paths and γ indicates the number of arcs from U in the UZ-Path.

The WV-Paths are nonempty, so $\eta_i \geq 1$ for all i. A WV-Path with η_i arcs from W has $\eta_i - 1$ arcs from V. Both of these arcs have length -1 in $D(A, \succ, \alpha, 1)$, so the WV-Paths have length $1 - 2\eta_i$. The UV-Path has length $\alpha\gamma$ since each U arc has length α and the arcs from Z have length 0. So

$$length(C) = \alpha\gamma + \sum_{i=1}^{k} (1 - 2\eta_i). \tag{13}$$

From Lemma 4.6, the cycle has length -1 or -2. If $length(C) = -1$ then, from (13), (11) holds. Completing the negative cycle C is an arc $(r_y, l_x) \in Z$. This corresponds to $x \sim y$, violating (10). In the case that $length(C) = -2$, by Lemma 4.6, α is odd, C has exactly one arc from U and exactly one maximal WV-Path. So $\gamma = 1$ and $k = 1$ and $x \sim \succ^{\eta_1} y$ holds in the order. Also, (13) with $\gamma = k = 1$ and $length(C) = -2$ gives $\eta_1 = (\alpha + 3)/2$. Then since $x \sim y$ (as arc $(r_y, l_x) \in Z$), (12) is violated.

Conversely, suppose that when α is even, (10) fails for some integral $\eta_i \geq 1$, $\gamma \geq 1$ satisfying (11), or if α is odd (10) fails for some integral $\eta_i \geq 1$, $\gamma \geq 1$ satisfying (11) or (12) fails. We show that $D(A, \succ, \alpha, 1)$ contains a negative cycle. By the correspondence between $x \sim^{\gamma} \succ^{\eta_1} \sim \succ^{\eta_2} \cdots \sim \succ^{\eta_k} y$ and P_0, P as described above, if a condition (10) is violated or if (12) is violated with $x \sim y$, then D contains a negative closed path, and hence a negative cycle. The closed path can be formed since $x \sim y$ implies that $(r_y, l_x) \in Z$. The negativity follows from (11) or from $\eta_1 = (\alpha + 3)/2$. In the case that (10) or (12) is violated with $y \succ x$, we proceed as in the proof of Theo-

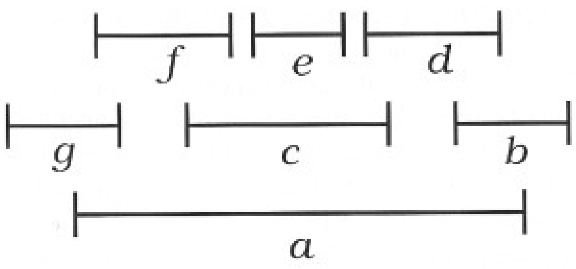


Fig. 2. An order with a duplicated element in a negative cycle.

rem 5.1. If $y > x$, then $(l_y, r_x) \in W$. Let P_0' be P_0 with the first arc (l_x, r_x) removed. Then $length(P_0') < length(P_0)$ and $C' = P_0', P, l_y, r_x$ has negative length. Either C' is itself a negative cycle or it contains a negative cycle. In either case, D contains a negative cycle. □

In both of the previous theorems, the elements of A appearing in the condition $x \sim >^{\eta_1} \sim \cdots >^{\eta_k} y$ are not necessarily distinct. For example, the order shown in Fig. 2 contains $a \sim b > c \sim d > e > f \sim c > g$ with $a \sim g$. Thus $a \sim > \sim >^2 \sim > g \Rightarrow a > g$ is violated. Since (9) holds for $\alpha = 3$, the order has no $[3,0]$ representation. The element c appears twice in the chain, corresponding to appearances as r_c and l_c in a negative cycle in the digraph. It can be checked that there is no condition (8) satisfying (9) (with $\alpha = 3$) which is violated that does not contain a repeated element.

In the case that degenerate intervals are allowed ($\beta = 0$), there are a finite number of conditions (8) which must be satisfied for each α since the η_i in (9) satisfy $1 \le \eta_i \le \alpha + 1$ and also k is at most $\alpha + 1$. It is not immediate that these conditions are independent. However, a more detailed description of the orders $(A, >) \in \mathscr{F}[\alpha, 0]$ given in Isaak [9] do imply that the set of conditions (8) satisfying (9) in Theorem 5.2 are independent. That is, for a given α and for each condition c defined by (8) and (9), there is an order which violates c but satisfies every other condition defined by (8) and (9).

In the case that only nondegenerate intervals are allowed ($\beta = 1$), for each γ and α there are a finite number of conditions (10) as k is bounded by $\alpha\gamma + 1$ in (11). However, for fixed α, the entire family of conditions described by (10) and (11) (and (12)) is infinite since γ may be any positive integer. The conditions in Theorem 5.3 are not independent. For example it can be shown that if (10) is violated for some η_1, $\gamma \ge 2$ and $k = 1$ satisfying (11) then a condition (10) satisfying (11) is violated with $\gamma = 1$ and $k = 1$ or (12) is violated. This is shown by a reduction of the corresponding cycles in the digraph. However, Theorem 5.5 shows that an infinite set of independent conditions is necessary to describe membership in $\mathscr{D}[\alpha, 1]$.

We now state results concerning the cardinality of the minimal families.

Theorem 5.4. *For a given $\alpha \ge 0$, $\mathscr{F}[\alpha, 0]$ is finite.*

Proof. If $(A, >) \in \mathscr{F}[\alpha, 0]$ then $(A, >)$ has no $[\alpha, 0]$ discrete representation. By Theorem 5.2, there is a subset of elements of A which violate (8) and satysfy (9) for some k and η_i. Also, as $(A, >)$ is minimal, each element of A must appear in the violated condition (8). Thus, the number of elements in A is bounded by $1 + \sum_{i=1}^{k} (\eta_i + 1) = \alpha + k + 2$, the number of elements appearing in a chain of the type in (8). Since $\sum_{i=1}^{k} \eta_i = \alpha + 1$ and the η_i are greater than or equal to one, k must satisfy $1 \le k \le \alpha + 1$. Thus, every order $(A, >)$ in $\mathscr{F}[\alpha, 0]$ satisfies $|A| \le \alpha + k + 2 \le 2\alpha + 3$. For a given α, there is a finite number of orders with at most $2\alpha + 3$ elements. So $\mathscr{F}[\alpha, 0]$ must be finite. □

For discrete representations in which the lower bound on interval length is one, the situation is quite different. There is no finite list of forbidden suborders to an $[\alpha,1]$ discrete representation. We give a simple example to prove this.

Theorem 5.5. *For $\alpha\geq 2$, $\mathscr{F}[\alpha,1]$ is infinite.*

Proof. For even α, construct an infinite family of minimal forbidden orders. A similar, slightly more complex construction, can be used for odd α. We will omit the construction for odd α. Details can be found in Isaak [9].

For α even, and any $\gamma\geq 3$, we will construct an order $(A^{\alpha,\gamma},\succ)$ on $(\alpha/2)+4+\gamma$ elements using an interval representation for which every interval except one has length between 1 and α. The exceptional interval has length $\alpha+1$. We then show that the corresponding digraph $D(A^{\alpha,\gamma},\succ,\alpha,1)$ contains a negative cycle, so there can be no $[\alpha,1]$ discrete representation. Finally we show that $(A^{\alpha,\gamma},\succ)$ is minimal by shifting the intervals to produce an $[\alpha,1]$ discrete representation for any suborder obtained by removing one element from $A^{\alpha,\gamma}$.

Let $A^{\alpha,\gamma}=\{a_1',a_2',a_3',a_4'\}\cup\{a_1,\dots,a_{(\alpha/2)\gamma+1}\}\cup\{b_1,\dots,b_\gamma\}$. Let an interval representation be given as follows.

$$\begin{aligned}
&J(a_1')=[0,1],\\
&J(a_2')=[2,4],\\
&J(a_3')=[2((\alpha/2)\gamma+1)-3,2((\alpha/2)\gamma+1)-1],\\
&J(a_4')=[2((\alpha/2)\gamma+1),2((\alpha/2)\gamma+1)+1],\\
&J(a_1)=[1,2],\\
&J(a_2)=[3,4],\\
&\ \vdots\\
&J(a_i)=[2i-1,2i],\\
&\ \vdots\\
&J(a_{(\alpha/2)\gamma+1})=[2((\alpha/2)\gamma+1)-1,2((\alpha/2)\gamma+1)],\\
&J(b_1)=[1,\alpha+2],\\
&J(b_2)=[\alpha+2,2\alpha+2],\\
&J(b_3)=[2\alpha+2,3\alpha+2],\\
&\ \vdots\\
&J(b_i)=[(i-1)\alpha+2,i\alpha+2],\\
&\ \vdots\\
&J(b_\gamma)=[(\gamma-1)\alpha+2,\gamma\alpha+2].
\end{aligned}$$

Note that $J(b_1)$ is the only interval with length greater than α. The intervals $J(b_i)$ for $i\neq 1$ can be shifted one unit to the left (i.e., $J(b_i)=[(i-1)\alpha+1,i\alpha+1]$) without changing the intersection relationship with the $J(a_j)$ intervals and the intervals $J(a_1')$, $J(a_2')$ and $J(a_3')$. Similarly, the $J(a_i)$ intervals can be shifted one unit right without changing the intersection relationship with the $J(b_i)$ intervals. See Fig. 3 for a schematic representation of $(A^{\alpha,\gamma},\succ)$. Note that when $\alpha=2$, the intervals for

$J(b_{\gamma-1})$ and $J(a_3')$ and for $J(b_2)$ and $J(a_2')$ should also overlap in this figure.

It is not difficult to check that the following are paths in $D(A^{\alpha,\gamma}, >, \alpha, 1)$.

$$P1 = l_{b_1}, r_{b_1}, l_{b_2}, r_{b_2}, \ldots, l_{b_\gamma}, r_{b_\gamma},$$

$$P2 = l_{a_4'}, r_{a_3'},$$

$$P3 = l_{a_{(\alpha/2)\gamma+1}}, r_{a_{(\alpha/2)\gamma}}, l_{a_{(\alpha/2)\gamma}}, r_{a_{(\alpha/2)\gamma-1}}, l_{a_{(\alpha/2)\gamma-1}}, \ldots, r_{a_2}, l_{a_2}, r_{a_1},$$

$$P4 = l_{a_2'}, r_{a_1'}, l_{b_1}.$$

Additionally $C = P1, P2, P3, P4$ is a cycle with the links between each pair of paths having length 0. We also have $length(P1) = \alpha\gamma$, $length(P2) = -1$, $length(P3) = -(\alpha\gamma - 1) = -\alpha\gamma + 1$, and $length(P4) = -1$. So the total length of the cycle is -1. Thus, by Theorem 3.2, $(A^{\alpha,\gamma}, >) \notin \mathscr{D}[\alpha, 1]$.

To show that $(A^{\alpha,\gamma}, >)$ is minimal, i.e., that each proper suborder of $(A^{\alpha,\gamma}, >)$ is in $\mathscr{D}[\alpha, 1]$, we construct an $[\alpha, 1]$ representation for each suborder obtained by deleting one element from $(A^{\alpha,\gamma}, >)$. If b_1 is removed, the above representation suffices. If some other element is removed, we shift the intervals for some of the elements in order to shorten the interval for b_1 without changing any of the relations. We give the shifts below; in each case it is not difficult to check that no overlaps of intervals are created or destroyed.

(i) Remove b_j for some $1 < j \le \gamma$: shrink $J(b_1)$ and for $i < j$ shift $J(b_i)$ one unit to the left;

$$J(b_i) = [(i-1)\alpha + 1, i\alpha + 1] \quad \text{for } 1 < i < j \quad \text{and} \quad J(b_1) = [1, \alpha + 1].$$

(ii) Remove a_j for some $3 \le j \le (\alpha/2)\gamma - 1$ (note that if $\alpha = 2$ and $\gamma = 3$ there is no such a_j): shrink $J(b_1)$ and $J(a_2')$ and shift $J(a_1')$ and $J(a_i)$ for $i < j$ one unit to the right;

$$J(a_1') = [1, 2] \quad \text{and} \quad J(a_2') = [3, 4],$$

$$J(a_i) = [2i, 2i+1] \quad \text{for } 1 \le i < j \quad \text{and} \quad J(b_1) = [2, \alpha + 2].$$

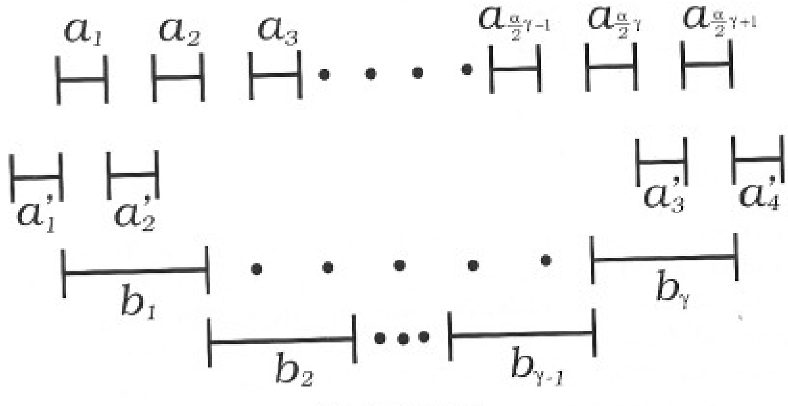


Fig. 3. $(A^{\alpha,\gamma}, >)$.

(iii) Remove a_1': shrink $J(b_1)$;

$$J(b_1) = [2, \alpha + 2].$$

(iv) Remove a_2': shrink $J(b_1)$ and shift $J(a_1')$ to the right;

$$J(a_1') = [1, 2] \quad \text{and} \quad J(b_1) = [2, \alpha + 2].$$

(v) Remove a_3': shrink $J(b_1)$, move $J(b_i)$ one unit to the left for $i = 2, \ldots, \gamma$ and move $J(a_4')$ one unit to the left;

$$J(b_i) = [(i-1)\alpha + 1, i\alpha + 1] \quad \text{for } i = 2, \ldots, \gamma \quad \text{and} \quad J(b_1) = [1, \alpha + 1],$$

$$J(a_4') = \left[2\left(\frac{\alpha}{2}\gamma + 1\right) - 1, 2\left(\frac{\alpha}{2}\gamma + 1\right)\right].$$

(vi) Remove a_4': shrink $J(b_1)$, and move $J(b_i)$ one unit to the left for $i = 2, \ldots, \gamma$;

$$J(b_i) = [(i-1)\alpha + 1, i\alpha + 1] \quad \text{for } i = 2, \ldots, \gamma \quad \text{and} \quad J(b_1) = [1, \alpha + 1].$$

(vii) Remove a_1: shrink $J(b_1)$ and $J(a_2')$ and shift $J(a_1')$ to the right;

$$J(a_1') = [1, 2] \quad \text{and} \quad J(a_2') = [3, 4] \quad \text{and} \quad J(b_1) = [2, \alpha + 2].$$

(viii) Remove a_2: shrink $J(b_1)$ and $J(a_2')$, and move $J(a_1')$ and $J(a_1)$ to the right;

$$J(a_1') = [1, 2] \quad \text{and} \quad J(a_2') = [3, 4] \quad \text{and} \quad J(a_1) = [2, 3] \quad \text{and} \quad J(b_1) = [2, \alpha + 2].$$

(ix) Remove $a_{(\alpha/2)\gamma}$: shrink $J(b_1)$ and $J(a_3')$, move $J(b_i)$ one unit to the left for $i = 2, \ldots, \gamma$, and move $J(a_4')$ and $J(a_{(\alpha/2)\gamma+1})$ one unit to the left;

$$J(b_i) = [(i-1)\alpha + 1, i\alpha + 1] \quad \text{for } i = 2, \ldots, \gamma \quad \text{and} \quad J(b_1) = [1, \alpha + 1],$$

$$J(a_3') = \left[2\left(\frac{\alpha}{2}\gamma + 1\right) - 3, 2\left(\frac{\alpha}{2}\gamma + 1\right) - 2\right] \quad \text{and}$$

$$J(a_4') = \left[2\left(\frac{\alpha}{2}\gamma + 1\right) - 1, 2\left(\frac{\alpha}{2}\gamma + 1\right)\right],$$

$$J(a_{(\alpha/2)\gamma+1}) = \left[2\left(\frac{\alpha}{2}\gamma + 1\right) - 2, 2\left(\frac{\alpha}{2}\gamma + 1\right) - 1\right].$$

(x) Remove $a_{(\alpha/2)\gamma+1}$: shrink $J(b_1)$ and $J(a_3')$, move $J(b_i)$ one unit to the left for $i = 2, \ldots, \gamma$, and move $J(a_4')$ one unit to the left;

$$J(b_i) = [(i-1)\alpha + 1, i\alpha + 1] \quad \text{for } i = 2, \ldots, \gamma \quad \text{and} \quad J(b_1) = [1, \alpha + 1],$$

$$J(a_3') = \left[2\left(\frac{\alpha}{2}\gamma + 1\right) - 3, 2\left(\frac{\alpha}{2}\gamma + 1\right) - 2\right] \quad \text{and}$$

$$J(a_4') = \left[2\left(\frac{\alpha}{2}\gamma + 1\right) - 1, 2\left(\frac{\alpha}{2}\gamma + 1\right)\right]. \qquad \square$$

Finally, we construct a special class of interval orders based on the violated condition (12) of Theorem 5.3.

Definition 5.6. Given α odd $\alpha \geq 3$, the *bi-minimal* order $(A, \succ)$ with respect to α is such that the elements can be labeled $A = \{a_0, a_1, \ldots, a_{(\alpha+5)/2}\}$ with $\succ$ given by $a_1 \succ a_2 \succ \cdots \succ a_{(\alpha+5)/2}$ (and the relations implied by transitivity in this chain) and for $i = 1, \ldots, (\alpha+5)/2$, $a_0 \sim a_i$.

Thus, the bi-minimal order with respect to α consists of a chain of $(\alpha+5)/2$ elements and a single element which is $\sim$ to every element in the chain. The bi-minimal order with respect to α has no $[\alpha+1, 1]$ discrete representation, but every proper suborder has an $[\alpha, 1]$ representation.

Theorem 5.7. *Given $\alpha \geq 3$, the bi-minimal order $(A, \succ)$ with respect to α satisfies $(A, \succ) \in \mathscr{F}[\alpha+1, 1]$ and $(A, \succ) \in \mathscr{F}[\alpha, 1]$.*

Proof. Let $\zeta = (\alpha+5)/2$ and let the elements of the bi-minimal order $(A, \succ)$ be labeled as in Definition 5.6. It can be checked that $C = l_{a_0}, r_{a_0}, l_{a_1}, r_{a_2}, l_{a_2}, \ldots, r_{a_{\zeta-1}}, l_{a_{\zeta-1}}, r_{a_\zeta}, l_{a_0}$ is a cycle in $D(A, \succ, \alpha+1, 1)$ with length -1. (The cycle contains one arc from U, two arcs from Z, and a WV-Path with $\zeta-1$ arcs from W and $\zeta-2$ arcs from V.) So $(A, \succ) \notin \mathscr{D}[\alpha+1, 1]$ (and thus $(A, \succ) \notin \mathscr{D}[\alpha, 1]$).

The proof will be completed by showing that for all $a \in A$, $(A \setminus \{a\}, \succ) \in \mathscr{D}[\alpha, 1]$ (and thus $(A \setminus \{a\}, \succ) \in \mathscr{D}[\alpha+1, 1]$).

Consider $A \setminus \{a_0\}$. The set of intervals with length 1 given by $J(a_i) = [\alpha+4-2i, \alpha+5-2i]$ for $i = 1, \ldots, (\alpha+5)/2$ can easily be seen to represent $(A \setminus \{a_0\}, \succ)$.

Consider $A \setminus \{a_j\}$ for a given $j \in \{1, \ldots, (\alpha+5)/2\}$. The set of intervals given by $J(a_0) = [0, \alpha]$ and

$$J(a_i) = \begin{cases} [\alpha+2-2i, \alpha+3-2i], & \text{if } i<j, \\ [\alpha+4-2i, \alpha+5-2i], & \text{if } i>j, \end{cases}$$

can easily be seen to represent $(A \setminus \{a_j\}, \succ)$. $\square$

Note that it is not difficult to construct an $[\alpha+2, 1]$ discrete representation for the bi-minimal order $(A, \succ)$ with respect to α by using the representation given in the proof for the case $A \setminus \{a_0\}$ along with the interval $J(a_0) = [0, \alpha+2]$. So $(A, \succ)$ has the property that it has an $[\alpha+2, 1]$ discrete representation, but no $[\alpha+1, 1]$ discrete representation, and every proper suborder has an $[\alpha, 1]$ discrete representation. So, by removing a single element, the length of the longest required interval is reduced by 2.

6. Conclusion

We may also consider bounded discrete representations of interval graphs. An *interval graph* G is a co-comparability graph of an interval order. That is, $\{v, w\}$ is an edge in G if and only if $v \sim w$ in the order. Alternatively, an interval graph has an interval representation in which $\{v, w\}$ is an edge if and only if the corresponding intervals overlap. In general, there may be several different interval orders for which a given interval graph is the co-comparability graph. As with interval orders, we will say that an $[\alpha, \beta]$ representation has intervals with lengths bounded by the vectors α and β. The representation is discrete if the endpoints are integers. Fishburn [4, Chapter 8; or 5] sketches a proof that an interval graph G has an $[\alpha, \beta]$ (nondiscrete, constant bounds) bounded representation if and only if every interval for which G is a co-comparability graph has such a representation. The same proof works for bounded discrete representations. Simply note that the transformations of given representations from one agreeing order to another can be done preserving integrality. In fact the proof works if variable upper bounds α are allowed.

With variable lower bounds, the comments in the preceding paragraph do not hold. Consider the interval graph G shown in Fig. 4(a). The discrete interval graph representation shown in Fig. 4(b) satisfies $1 \le |J(a)| \le 3$, $2 \le |J(b)| \le 3$, $1 \le |J(c)| \le 3$, and $1 \le |J(d)| \le 3$. Here $|J(i)|$ indicates the length of interval $J(i)$. The interval order shown in Fig. 4(c) (via its Hasse digram) has G as a co-comparability graph and also has the same representation satisfying the same bounds. However, the interval order shown in Fig. 4(d), which also has G as a co-comparability graph, has no representa-

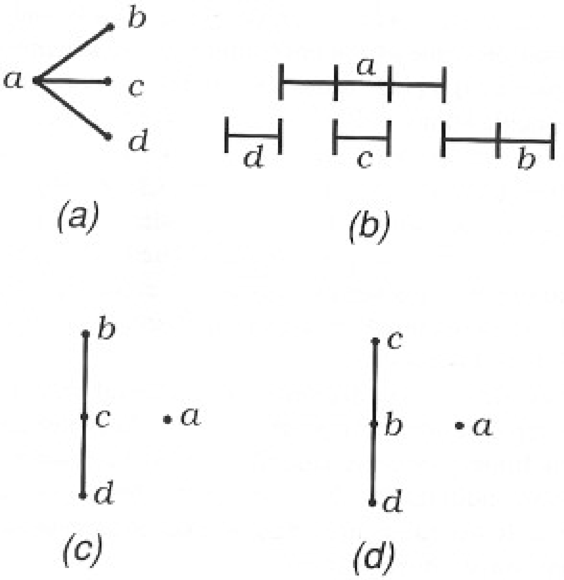


Fig. 4. (a) An interval graph. (b) A discrete interval graph representation. (c) and (d) Two agreeing orders.

tion satisfying the bounds stated above. In this case, if the intervals for c and d both intersect the interval for a, there are at most two integers between the left endpoint of c and the right endpoint d. Then the interval for b, which must fall between these endpoints can have length at most 1, violating the lower bound for b. Thus, the algorithmic result which we have given for interval order representations does not carry over in general to interval graphs. It would be interesting to find an algorithm for determining if an interval graph has an $[\alpha, \beta]$ discrete representation given the nonconstant bounds α, β.

Fishburn and Graham [6] examine graphs which have $[\alpha, 1]$ (nondiscrete) representations, and for rational α, families of minimal graphs with no $[\alpha, 1]$ representation, but an $[\alpha', 1]$ representation for $\alpha' > \alpha$. They obtain general bounds on the size of these minimal families and exact counts in certain special cases.

We may use the negative cycles in the corresponding digraph in the nondiscrete case to remove one of the conditions in Fishburn's theorem [3] stating necessary and sufficient conditions in the case of nondiscrete bounded orders. Fishburn notes that, in the nondiscrete case, by scaling, we may assume that the bounds are relatively prime integers. His theorem states that an interval order has representation with interval lengths bounded between relatively prime positive integers p and q $(p \leq q)$ if and only if

$$x >^{\zeta_1} \sim^{\xi_1} \ldots >^{\zeta_n} \sim^{\xi_n} y \quad \Rightarrow \quad x > y$$

and

$$x \sim^{\xi_n} >^{\zeta_n} \ldots \sim^{\xi_1} >^{\zeta_1} y \quad \Rightarrow \quad x > y$$

for $n = 1, \ldots, p$, $(\zeta_1, \xi_1, \ldots, \zeta_n, \xi_n) \geq (2, 2, \ldots, 2, 1)$, $\sum_{i=1}^{n} \zeta_i = q + n$, and $\sum_{i=1}^{n} \xi_i = p + n - 1$.

We can show that only one of the implications in the statement of this theorem is needed. Construct a digraph D using the nondiscrete conditions as described in Remark 3.4. The implications $x >^{\zeta_1} \sim^{\xi_1} \ldots >^{\zeta_n} \sim^{\xi_n} y \Rightarrow x > y$ together with $y \sim x$ correspond to a negative cycle in D. If the implications are violated, there is such a negative cycle. Breaking the cycle with an arc $(r_y, l_x) \in Z$ such that the next arc in the cycle is $(l_x, r_x) \in U$ (such an arc exists by Corollary 4.2) produces a set of implications $x \sim^{\xi_n} >^{\zeta_n} \ldots \sim^{\xi_1} >^{\zeta_1} y \Rightarrow x > y$ which are violated. The converse is shown in a similar manner, noting that any negative cycle must contain an arc from Z followed by an arc from W (where we break the cycle), since otherwise the cycle contains only arcs from U and V and is positive.

We have given an algorithm to determine if an interval order $(A, >)$ has an $[\alpha, \beta]$ bounded discrete representation for general bounds. We have also given necessary and sufficient conditions for representability when the bounds are constant. We have more succinct conditions for the cases that the lower bound is the constant 0 and the constant 1. It would be interesting to examine similar succinct conditions when the constant lower bound is $\beta \geq 2$.

So far we have examined finite interval orders. We might ask about bounded representations of infinite interval orders which have real representations. Given such an order that has no $[\alpha, \beta]$ bounded discrete representation, is there a finite suborder that also has no bounded discrete representation?

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References

[1] K.P. Bogart and K. Stellpflug, Discrete representation theory for semiorders I: Fundamental examples, Manuscript (1989).

[2] K.P. Bogart and K. Stellpflug, Discrete representation theory for semiorders II: Representation theorems, Manuscript (1990).

[3] P.C. Fishburn, Threshold-bounded interval orders and a theory of picycles, SIAM J. Algebraic Discrete Methods 4 (1983) 290–305.

[4] P.C. Fishburn, Interval Orders and Interval Graphs (Wiley, New York, 1985).

[5] P.C. Fishburn, Interval graphs and interval orders, Discrete Math. 55 (1985) 135–149.

[6] P.C. Fishburn and R.L. Graham, Classes of interval graphs under expanding length restrictions, J. Graph. Theory 9 (1985) 459–472.

[7] P.C. Fishburn and B. Monjardet, Norbert Wiener on the theory of measurement, Manuscript (1990).

[8] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).

[9] G. Isaak, Odd forests, reversing numbers, and discrete representations of interval orders, Ph.D. Thesis, RUTCOR (Rutgers Center for Operations Research), Rutgers University, New Brunswick, NJ (1990).

[10] E. Lawler, Combinatorial Optimization: Networks and Matroids (Holt, Rinehart, and Winston, New York, 1976).

[11] F.S. Roberts, Measurement Theory (Cambridge University Press, Cambridge, 1979).

[12] A. Schrijver, Theory of Integer and Linear Programming (Wiley, New York, 1986).

[13] N. Wiener, A conbribution to the theory of relative position, Trans. Amer. Math. Soc. 5 (1914) 343–384.