When does
$$\begin{aligned} x &\leq a \\ x &\geq 0 \end{aligned}$$
have a solution?

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When does $\begin{array}{rcl} x & \leq & a \\ x & \geq & 0 \end{array}$ have a solution?

When does $\begin{array}{rcl} x & \leq & a \\ x & \geq & 0 \end{array}$ have a solution?

When does a system of linear inequalities have a solution?

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When does a system of pipelines have a feasible flow?

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When does a system of linear inequalities have a solution?

When does a system of pipelines have a feasible flow?

When does an order relation represent ‘comes before’ for intervals in time?

When does $\begin{array}{rcl} x & \leq & a \\ x & \geq & 0 \end{array}$ have a solution?

When does a system of linear inequalities have a solution?

When does a system of pipelines have a feasible flow?

When does an order relation represent ‘comes before’ for intervals in time?

When does a list of numbers represent the win records for a round- robin tournament?

When does $\begin{array}{rcl} x & \leq & a \\ x & \geq & 0 \end{array}$ have a solution?

When does $\begin{array}{ll} x & \leq a \\ x & \geq 0 \end{array}$ have a solution?

The answer should be 'obvious' but try an example:

When does $\begin{matrix} x & \leq & a \\ x & \geq & 0 \end{matrix}$ have a solution?

The answer should be ‘obvious’ but try an example:

$$\begin{matrix} x & \leq & 13 \\ x & \geq & 0 \end{matrix} \qquad \begin{matrix} x & \leq & -42 \\ x & \geq & 0 \end{matrix}$$

When does $\begin{array}{lcl} x & \leq & a \\ x & \geq & 0 \end{array}$ have a solution?

The answer should be 'obvious' but try an example:

$$\begin{array}{lcr} x & \leq & 13 \\ x & \geq & 0 \end{array} \qquad\qquad \begin{array}{lcr} x & \leq & -42 \\ x & \geq & 0 \end{array}$$

Has a solution
for example $x = 7$

When does $\begin{array}{lcl} x & \leq & a \\ x & \geq & 0 \end{array}$ have a solution?

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Has no solution
Why not?

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Has a solution
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Has no solution
Why not?

If, for example x^* solves $\begin{array}{rcr} x & \leq & -42 \\ x & \geq & 0 \end{array}$

When does $\begin{array}{rcl} x & \le & a \\ x & \ge & 0 \end{array}$ have a solution?

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$$\begin{array}{rcr} x & \le & 13 \\ x & \ge & 0 \end{array} \qquad\qquad \begin{array}{rcr} x & \le & -42 \\ x & \ge & 0 \end{array}$$

Has a solution
for example $x = 7$

Has no solution
Why not?

If, for example x^* solves $\begin{array}{rcr} x & \le & -42 \\ x & \ge & 0 \end{array}$

Then $0 \le x^* \le -42$

When does $\begin{array}{ccc} x & \leq & a \\ x & \geq & 0 \end{array}$ have a solution?

The answer should be ‘obvious’ but try an example:

$$\begin{array}{ccc} x & \leq & 13 \\ x & \geq & 0 \end{array}$$

Has a solution
for example $x = 7$

$$\begin{array}{ccc} x & \leq & -42 \\ x & \geq & 0 \end{array}$$

Has no solution
Why not?

If, for example x^* solves $\begin{array}{ccc} x & \leq & -42 \\ x & \geq & 0 \end{array}$

Then $0 \leq x^* \leq -42 \Rightarrow 0 \leq -42.$

When does $\begin{matrix} x & \leq & a \\ x & \geq & 0 \end{matrix}$ have a solution?

The answer should be 'obvious' but try an example:

$$\begin{matrix} x & \leq & 13 \\ x & \geq & 0 \end{matrix}$$

Has a solution
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$$\begin{matrix} x & \leq & -42 \\ x & \geq & 0 \end{matrix}$$

Has no solution
Why not?

If, for example x^* solves $\begin{matrix} x & \leq & -42 \\ x & \geq & 0 \end{matrix}$

Then $0 \leq x^* \leq -42 \Rightarrow 0 \leq -42.$

So there is no solution

Generalize to systems of linear inequalities

$$\begin{aligned} x_1 + 4x_2 - x_3 &\le 2 \\ -2x_1 - 3x_2 + x_3 &\le 1 \\ -3x_1 - 2x_2 + x_3 &\le 0 \\ 4x_1 + x_2 - x_3 &\le -1 \end{aligned}$$

$$\begin{aligned} x_1 + 4x_2 - x_3 &\le 1 \\ -2x_1 - 3x_2 + x_3 &\le -2 \\ -3x_1 - 2x_2 + x_3 &\le 1 \\ 4x_1 + x_2 - x_3 &\le 1 \end{aligned}$$

Generalize to systems of linear inequalities

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Has a solution
for example $x_1 = 0, x_2 = 1, x_3 = 2$

Generalize to systems of linear inequalities

$$\begin{aligned} x_1 + 4x_2 - x_3 &\le 2 \\ -2x_1 - 3x_2 + x_3 &\le 1 \\ -3x_1 - 2x_2 + x_3 &\le 0 \\ 4x_1 + x_2 - x_3 &\le -1 \end{aligned}$$

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$$\begin{aligned} x_1 + 4x_2 - x_3 &\le 1 \\ -2x_1 - 3x_2 + x_3 &\le -2 \\ -3x_1 - 2x_2 + x_3 &\le 1 \\ 4x_1 + x_2 - x_3 &\le 1 \end{aligned}$$

Has no solution

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Has a solution
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$$\begin{aligned} x_1 + 4x_2 - x_3 &\leq 1 \\ -2x_1 - 3x_2 + x_3 &\leq -2 \\ -3x_1 - 2x_2 + x_3 &\leq 1 \\ 4x_1 + x_2 - x_3 &\leq 1 \end{aligned}$$

Has no solution
Why not?

Show

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & x_3 & \leq & 1 \\ -2x_1 & - & 3x_2 & + & x_3 & \leq & -2 \\ -3x_1 & - & 2x_2 & + & x_3 & \leq & 1 \\ 4x_1 & + & x_2 & - & x_3 & \leq & 1 \end{array}$$

Has no solution

Show

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & x_3 & \leq & 1 \\ -2x_1 & - & 3x_2 & + & x_3 & \leq & -2 \\ -3x_1 & - & 2x_2 & + & x_3 & \leq & 1 \\ 4x_1 & + & x_2 & - & x_3 & \leq & 1 \end{array}$$

Has no solution

$$\begin{array}{rl} 1\ (& x_1 + 4x_2 - x_3 \leq 1) \\ 3(& -2x_1 - 3x_2 + x_3 \leq -2) \\ -3\ (& -3x_1 - 2x_2 + x_3 \leq 1) \\ -1\ (& 4x_1 + x_2 - x_3 \leq 1) \end{array}$$

Show

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & x_3 & \leq & 1 \\ -2x_1 & - & 3x_2 & + & x_3 & \leq & -2 \\ -3x_1 & - & 2x_2 & + & x_3 & \leq & 1 \\ 4x_1 & + & x_2 & - & x_3 & \leq & 1 \end{array}$$

Has no solution

$$\begin{array}{rl} 1\,(& x_1 + 4x_2 - x_3 \leq 1) \\ 3(& -2x_1 - 3x_2 + x_3 \leq -2) \\ -3\,(& -3x_1 - 2x_2 + x_3 \leq 1) \\ -1\,(& 4x_1 + x_2 - x_3 \leq 1) \end{array} \quad \Rightarrow$$

Show

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & x_3 & \leq & 1 \\ -2x_1 & - & 3x_2 & + & x_3 & \leq & -2 \\ -3x_1 & - & 2x_2 & + & x_3 & \leq & 1 \\ 4x_1 & + & x_2 & - & x_3 & \leq & 1 \end{array}$$

Has no solution

$$\begin{array}{r} 1\,(\quad x_1 + 4x_2 - x_3 \leq 1) \\ 3(-2x_1 - 3x_2 + x_3 \leq -2) \\ -3\,(-3x_1 - 2x_2 + x_3 \leq 1) \\ -1\,(\quad 4x_1 + x_2 - x_3 \leq 1) \end{array} \Rightarrow \begin{array}{r} x_1 + 4x_2 - x_3 \leq 1 \\ -6x_1 - 9x_2 + 3x_3 \leq -6 \\ 9x_1 + 6x_2 - 3x_3 \leq -3 \\ -4x_1 - x_2 + x_3 \leq -1 \\ \hline 0x_1 + 0x_2 + 0x_3 \leq -9 \end{array}$$

Show

$$\begin{array}{rrrcr} x_1 & + 4x_2 & - x_3 & \leq & 1 \\ -2x_1 & - 3x_2 & + x_3 & \leq & -2 \\ -3x_1 & - 2x_2 & + x_3 & \leq & 1 \\ 4x_1 & + x_2 & - x_3 & \leq & 1 \end{array}$$

Has no solution

$$\begin{array}{r} 1\,(\quad x_1 + 4x_2 - x_3 \leq 1) \\ 3(-2x_1 - 3x_2 + x_3 \leq -2) \\ -3\,(-3x_1 - 2x_2 + x_3 \leq 1) \\ -1\,(\quad 4x_1 + x_2 - x_3 \leq 1) \end{array} \Rightarrow \begin{array}{r} x_1 + 4x_2 - x_3 \leq 1 \\ -6x_1 - 9x_2 + 3x_3 \leq -6 \\ 9x_1 + 6x_2 - 3x_3 \leq -3 \\ -4x_1 - x_2 + x_3 \leq -1 \\ \hline 0x_1 + 0x_2 + 0x_3 \leq -9 \end{array}$$

What is wrong?

Show

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & x_3 & \leq & 1 \\ -2x_1 & - & 3x_2 & + & x_3 & \leq & -2 \\ -3x_1 & - & 2x_2 & + & x_3 & \leq & 1 \\ 4x_1 & + & x_2 & - & x_3 & \leq & 1 \end{array}$$

Has no solution

$$\begin{array}{rl} 1\,(& x_1+4x_2-x_3 \leq 1) \\ 3(& -2x_1-3x_2+x_3 \leq -2) \\ -3\,(& -3x_1-2x_2+x_3 \leq 1) \\ -1\,(& 4x_1+x_2-x_3 \leq 1) \end{array} \Rightarrow \begin{array}{r} x_1+4x_2-x_3 \leq 1 \\ -6x_1-9x_2+3x_3 \leq -6 \\ 9x_1+6x_2-3x_3 \leq -3 \\ -4x_1-x_2+x_3 \leq -1 \\ \hline 0x_1+0x_2+0x_3 \leq -9 \end{array}$$

What is wrong?

Multiplying by negatives changes direction of inequality

Show

$$\begin{aligned} x_1 + 4x_2 - x_3 &\leq 1 \\ -2x_1 - 3x_2 + x_3 &\leq -2 \\ -3x_1 - 2x_2 + x_3 &\leq 1 \\ 4x_1 + x_2 - x_3 &\leq 1 \end{aligned}$$

Has no solution

Show

$$\begin{aligned} x_1 + 4x_2 - x_3 &\leq 1 \\ -2x_1 - 3x_2 + x_3 &\leq -2 \\ -3x_1 - 2x_2 + x_3 &\leq 1 \\ 4x_1 + x_2 - x_3 &\leq 1 \end{aligned}$$

Has no solution

$$\begin{aligned} 3\,(\quad x_1 + 4x_2 - x_3 &\leq 1) \\ 4\,(-2x_1 - 3x_2 + x_3 &\leq -2) \\ 1\,(-3x_1 - 2x_2 + x_3 &\leq 1) \\ 2\,(\quad 4x_1 + x_2 - x_3 &\leq 1) \end{aligned}$$

Show

$$\begin{aligned} x_1 + 4x_2 - x_3 &\leq 1 \\ -2x_1 - 3x_2 + x_3 &\leq -2 \\ -3x_1 - 2x_2 + x_3 &\leq 1 \\ 4x_1 + x_2 - x_3 &\leq 1 \end{aligned}$$

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Show

$$\begin{aligned} x_1 + 4x_2 - x_3 &\le 1 \\ -2x_1 - 3x_2 + x_3 &\le -2 \\ -3x_1 - 2x_2 + x_3 &\le 1 \\ 4x_1 + x_2 - x_3 &\le 1 \end{aligned}$$

Has no solution

$$\begin{aligned} 3\,(\ x_1 + 4x_2 - x_3 &\le 1) \\ 4\,(-2x_1 - 3x_2 + x_3 &\le -2) \\ 1\,(-3x_1 - 2x_2 + x_3 &\le 1) \\ 2\,(\ 4x_1 + x_2 - x_3 &\le 1) \end{aligned} \Rightarrow \begin{aligned} 3x_1 + 12x_2 - 3x_3 &\le 3 \\ -8x_1 - 12x_2 + 4x_3 &\le -8 \\ -3x_1 - 2x_2 + x_3 &\le 1 \\ 8x_1 + 2x_2 - 2x_3 &\le 2 \\ \hline 0x_1 + 0x_2 + 0x_3 &\le -2 \end{aligned}$$

Show

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & x_3 & \le & 1 \\ -2x_1 & - & 3x_2 & + & x_3 & \le & -2 \\ -3x_1 & - & 2x_2 & + & x_3 & \le & 1 \\ 4x_1 & + & x_2 & - & x_3 & \le & 1 \end{array}$$

Has no solution

$$\begin{array}{l} 3\,(\;\;x_1 + 4x_2 - x_3 \le 1) \\ 4\,(-2x_1 - 3x_2 + x_3 \le -2) \\ 1\,(-3x_1 - 2x_2 + x_3 \le 1) \\ 2\,(\;\;4x_1 + x_2 - x_3 \le 1) \end{array} \Rightarrow \begin{array}{rcrcrcr} 3x_1 & + & 12x_2 & - & 3x_3 & \le & 3 \\ -8x_1 & - & 12x_2 & + & 4x_3 & \le & -8 \\ -3x_1 & - & 2x_2 & + & x_3 & \le & 1 \\ 8x_1 & + & 2x_2 & - & 2x_3 & \le & 2 \\ \hline 0x_1 & + & 0x_2 & + & 0x_3 & \le & -2 \end{array}$$

There is no solution

Show

$$\begin{array}{rcr} x_1 + 4x_2 - x_3 & \leq & 1 \\ -2x_1 - 3x_2 + x_3 & \leq & -2 \\ -3x_1 - 2x_2 + x_3 & \leq & 1 \\ 4x_1 + x_2 - x_3 & \leq & 1 \end{array}$$

Has no solution

$$\begin{array}{l} 3\,(\quad x_1 + 4x_2 - x_3 \leq 1) \\ 4\,(-2x_1 - 3x_2 + x_3 \leq -2) \\ 1\,(-3x_1 - 2x_2 + x_3 \leq 1) \\ 2\,(\quad 4x_1 + x_2 - x_3 \leq 1) \end{array} \Rightarrow \begin{array}{rcr} 3x_1 + 12x_2 - 3x_3 & \leq & 3 \\ -8x_1 - 12x_2 + 4x_3 & \leq & -8 \\ -3x_1 - 2x_2 + x_3 & \leq & 1 \\ 8x_1 + 2x_2 - 2x_3 & \leq & 2 \\ \hline 0x_1 + 0x_2 + 0x_3 & \leq & -2 \end{array}$$

There is no solution

$y_1 = 3,\ y_2 = 4,\ y_3 = 1,\ y_4 = 2$ is a certificate of inconsistency

$$
\begin{array}{rcr}
x_1 + 4x_2 - x_3 & \leq & 1 \\
-2x_1 - 3x_2 + x_3 & \leq & -2 \\
-3x_1 - 2x_2 + x_3 & \leq & 1 \\
4x_1 + x_2 - x_3 & \leq & 1
\end{array}
$$

$$
\begin{array}{ll}
3\,(& x_1 + 4x_2 - x_3 \leq 1 \,) \\
4\,(& -2x_1 - 3x_2 + x_3 \leq -2 \,) \\
1\,(& -3x_1 - 2x_2 + x_3 \leq 1 \,) \\
2\,(& 4x_1 + x_2 - x_3 \leq 1 \,) \\
\hline
 & 0x_1 + 0x_2 + 0x_3 \leq -2
\end{array}
$$

$$
\begin{pmatrix} 1 & 4 & -1 \\ -2 & -3 & 1 \\ -3 & -2 & 1 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}
$$

$$
(y_1\, y_2\, y_3\, y_4) \begin{pmatrix} 1 & 4 & -1 \\ -2 & -3 & 1 \\ -3 & -2 & 1 \\ 4 & 1 & -1 \end{pmatrix} = (0000)
$$

$$
(y_1\, y_2\, y_3\, y_4) \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix} < 0
$$

$$
(y_1\, y_2\, y_3\, y_4) \geq (0000)
$$

$$
A\mathbf{x} \leq \mathbf{b}
$$

$$
\mathbf{y}A = \mathbf{0}, \mathbf{y}\mathbf{b} < 0, \mathbf{y} \geq \mathbf{0}
$$

$$\begin{aligned} x_1 + 4x_2 - x_3 &\le 1 \\ -2x_1 - 3x_2 + x_3 &\le -2 \\ -3x_1 - 2x_2 + x_3 &\le 1 \\ 4x_1 + x_2 - x_3 &\le 1 \end{aligned}$$

$$\begin{aligned} 3\,(\quad x_1 + 4x_2 - x_3 &\le 1) \\ 4\,(-2x_1 - 3x_2 + x_3 &\le -2) \\ 1\,(-3x_1 - 2x_2 + x_3 &\le 1) \\ 2\,(\quad 4x_1 + x_2 - x_3 &\le 1) \\ \hline 0x_1 + 0x_2 + 0x_3 &\le -2 \end{aligned}$$

$$\begin{pmatrix} 1 & 4 & -1 \\ -2 & -3 & 1 \\ -3 & -2 & 1 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

$$(y_1\, y_2\, y_3\, y_4) \begin{pmatrix} 1 & 4 & -1 \\ -2 & -3 & 1 \\ -3 & -2 & 1 \\ 4 & 1 & -1 \end{pmatrix} = (0000)$$

$$(y_1\, y_2\, y_3\, y_4) \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix} < 0$$

$$(y_1\, y_2\, y_3\, y_4) \ge (0000)$$

$$A\mathbf{x} \le \mathbf{b} \qquad \mathbf{y}A = \mathbf{0}, \mathbf{y}\mathbf{b} < 0, \mathbf{y} \ge \mathbf{0}$$

At most one of these has a solution

$$\begin{array}{rcr} x_1 + 4x_2 - x_3 & \leq & 1 \\ -2x_1 - 3x_2 + x_3 & \leq & -2 \\ -3x_1 - 2x_2 + x_3 & \leq & 1 \\ 4x_1 + x_2 - x_3 & \leq & 1 \end{array}$$

$$\begin{array}{ll} 3\,(& x_1 + 4x_2 - x_3 \leq 1) \\ 4\,(& -2x_1 - 3x_2 + x_3 \leq -2) \\ 1\,(& -3x_1 - 2x_2 + x_3 \leq 1) \\ 2\,(& 4x_1 + x_2 - x_3 \leq 1) \\ \hline & 0x_1 + 0x_2 + 0x_3 \leq -2 \end{array}$$

$$\begin{pmatrix} 1 & 4 & -1 \\ -2 & -3 & 1 \\ -3 & -2 & 1 \\ 4 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

$$(y_1\, y_2\, y_3\, y_4) \begin{pmatrix} 1 & 4 & -1 \\ -2 & -3 & 1 \\ -3 & -2 & 1 \\ 4 & 1 & -1 \end{pmatrix} = (0000)$$

$$(y_1\, y_2\, y_3\, y_4) \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix} < 0$$

$$(y_1\, y_2\, y_3\, y_4) \geq (0000)$$

$$A\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{y}A = \mathbf{0}, \mathbf{y}\mathbf{b} < 0, \mathbf{y} \geq \mathbf{0}$$

At most one of these has a solution

In fact exactly one has a solution

Farkas' Lemma (1906)

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} \leq \mathbf{b} \qquad \text{II: } \mathbf{y}A = \mathbf{0}, \mathbf{y}\mathbf{b} < 0, \mathbf{y} \geq \mathbf{0}$$

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Equivalently (exercise - show this):

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \qquad \text{II: } \mathbf{y}A \geq \mathbf{0}, \mathbf{y}\mathbf{b} < 0$$

Farkas' Lemma (1906)

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} \leq \mathbf{b} \qquad \text{II: } \mathbf{y}A = \mathbf{0}, \mathbf{y}\mathbf{b} < 0, \mathbf{y} \geq \mathbf{0}$$

Equivalently (exercise - show this):

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \qquad \text{II: } \mathbf{y}A \geq \mathbf{0}, \mathbf{y}\mathbf{b} < 0$$

Compare to result from basic linear algebra (via Gaussian elimination for example):

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} = \mathbf{b} \qquad \text{II: } \mathbf{y}A = \mathbf{0}, \mathbf{y}\mathbf{b} \neq 0$$

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \qquad \text{II: } \mathbf{y}A \geq \mathbf{0}, \mathbf{yb} < 0$$

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \qquad \text{II: } \mathbf{y}A \geq \mathbf{0}, \mathbf{yb} < 0$$

Show (again, this time using matrix notation and associativity) that at most one has a solution:

$0 = \mathbf{0x} \leq (\mathbf{y}A)\,\mathbf{x} = \mathbf{y}\,(A\mathbf{x}) = \mathbf{yb} < 0$

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \qquad \text{II: } \mathbf{y}A \geq \mathbf{0}, \mathbf{y}\mathbf{b} < 0$$

Show (again, this time using matrix notation and associativity) that at most one has a solution:

$0 = \mathbf{0}\mathbf{x} \leq (\mathbf{y}A)\,\mathbf{x} = \mathbf{y}\,(A\mathbf{x}) = \mathbf{y}\mathbf{b} < 0$

Can be proved using Fourier-Motzkin elimination and mathematical induction or by using methods from linear programming

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \qquad \text{II: } \mathbf{y}A \geq \mathbf{0}, \mathbf{yb} < 0$$

Show (again, this time using matrix notation and associativity) that at most one has a solution:

$0 = \mathbf{0x} \leq (\mathbf{y}A)\,\mathbf{x} = \mathbf{y}\,(A\mathbf{x}) = \mathbf{yb} < 0$

Can be proved using Fourier-Motzkin elimination and mathematical induction or by using methods from linear programming

Fourier-Motzkin elimination:

- Separate inequalities into upper and lower bounds on a variable x
- Take all lower bound/upper bound pairs along with inequalities omitting x
- A solution to the new system yields a solution to the original; a certificate of inconsistency for the new system yields a certificate for the original
- inefficient by hand or on computer but a nice mathematical induction proof

Exactly one of the following has a solution:

$$\text{I: } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \qquad \text{II: } \mathbf{y}A \geq \mathbf{0}, \mathbf{y}\mathbf{b} < 0$$

Show (again, this time using matrix notation and associativity) that at most one has a solution:

$0 = \mathbf{0}\mathbf{x} \leq (\mathbf{y}A)\,\mathbf{x} = \mathbf{y}\,(A\mathbf{x}) = \mathbf{y}\mathbf{b} < 0$

Can be proved using Fourier-Motzkin elimination and mathematical induction or by using methods from linear programming

Fourier-Motzkin elimination:

- Separate inequalities into upper and lower bounds on a variable x
- Take all lower bound/upper bound pairs along with inequalities omitting x
- A solution to the new system yields a solution to the original; a certificate of inconsistency for the new system yields a certificate for the original
- inefficient by hand or on computer but a nice mathematical induction proof

There are practical algorithms for solving these as special instances of linear programming problems; big news when a 'new' 'efficient'

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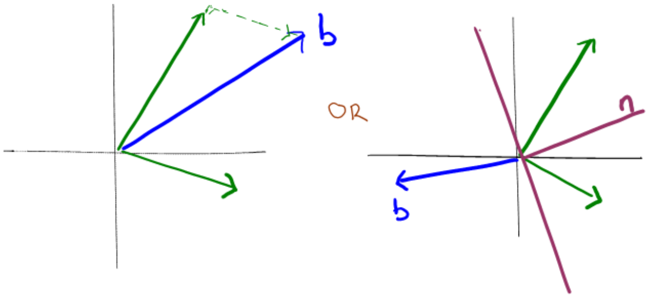
Geometric 'Proof' Either **b** is in the cone generated by the columns of A or there is a separating hyperplane with normal vector forming a and angle at most 90 degrees with the columns of A and greater than 90 degrees with **b**



Score Sequences of Round Robin Tournaments

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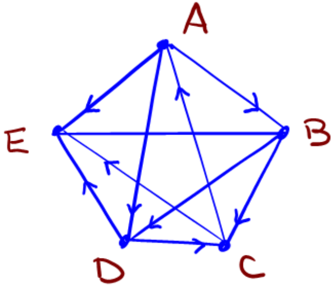
A wins 3 games, B wins 3 games, C wins 2 games, D wins 2 games, E wins 0 games

Score sequence is (3,3,2,2,0)



Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

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Use mathematical tools to make the check faster

For k players there are $\frac{n(n-1)}{2}$ games in a round robin tournament

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Which of the following are score sequences for a tournament with 7 players?

$(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$

$(5, 4, 3, 3, 3, 1, 0)$

$(3, 3, 3, 3, 3, 3, 3)$

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Landau (1951) considered tournaments in the context of pecking order in poultry populations

A necessary condition for a sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers to be the score sequence of a round-robin tournament:

$$\sum_{i \in I} s_i \geq \frac{|I|(|I|-1)}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\}$$

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The number of wins for any set of teams must be as large as the number of games played between those teams
and
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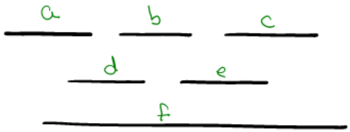
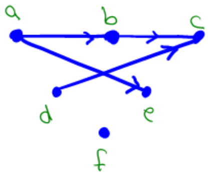
The sequence
$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$
can be checked by hand in a few minutes. It is not a score sequence

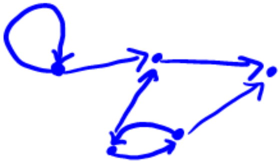
Representing Intervals in Time

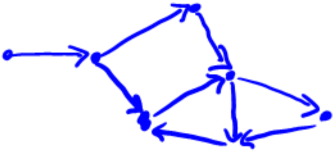
A set of intervals and an interval digraph representation:
The arcs represent ‘comes before’ in time
(arcs implied by transitivity are not shown)

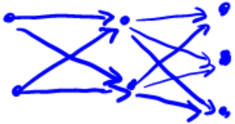


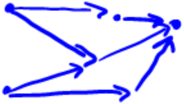


Which of the following are interval digraphs representing 6 intervals?









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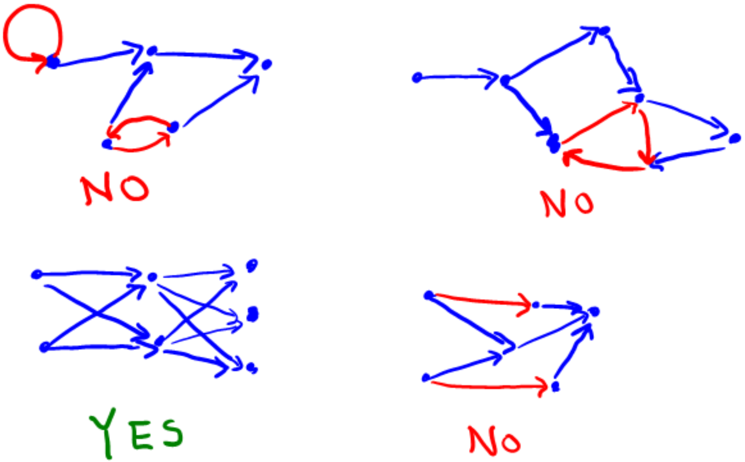
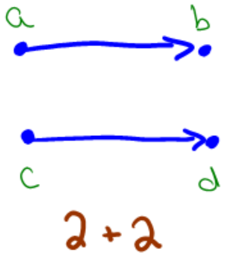


An interval digraph cannot contain a $\mathbf{2} + \mathbf{2}$



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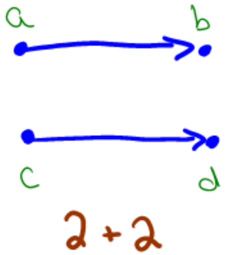

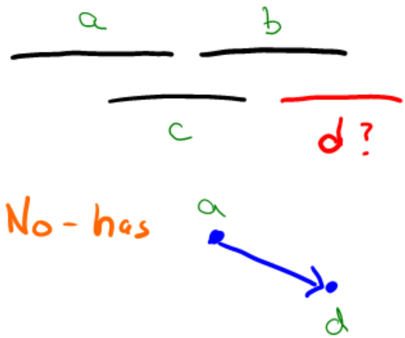

Weiner (1915) considered representations of intervals in time, Benzer (1959) considered intervals as representations of intervals formed by gene splices, Fishburn (1970) considered interval digraphs representing intransitive indifference in preference relations, other applications include seriation in archeology, scheduling etc.

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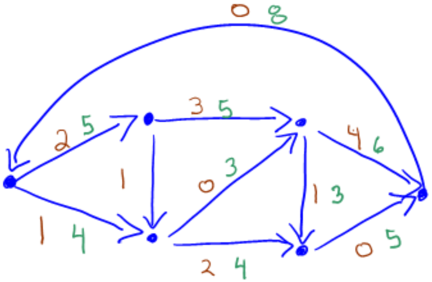
Circulations

Circulations

A digraph with upper and lower flow bounds on a possible circulation:

A circulation satisfies flow conservation



Circulations

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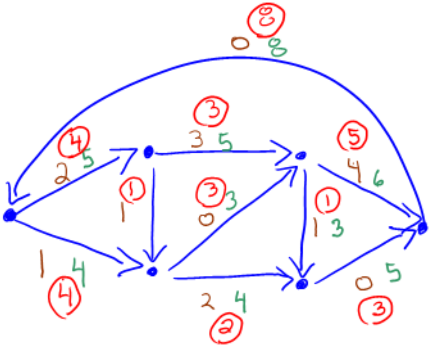

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This condition can be formalized as: If a digraph along with upper and lower bounds $u(xy)$ and $l(xy)$ has a circulation then

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Hoffman's Circulation Theorem (1956): This necessary condition is also sufficient

Farkas' Lemma (1906):

Exactly one of the following has a solution:

I: $A\mathbf{x} \leq \mathbf{b}$ II: $\mathbf{y}A = \mathbf{0}, \mathbf{y}\mathbf{b} < 0, \mathbf{y} \geq \mathbf{0}$

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A sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers is the score sequence of a round-robin tournament if and only if:

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What do Farkas' Lemma, Landau's Theorem, Fishburn's Theorem and Hoffman's Circulation Theorem have in common?

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All can be viewed as instances of:

Either a system of linear inequalities has a solution or it is inconsistent

Landau's Theorem via systems of linear inequalities

- Possible score sequence $(s_1, s_2, \ldots, s_n)$
- For each integral pair $1 \leq i < j \leq n$ define a variable $x_{i,j}$ with $x_{i,j} = 1$ if i beats j and $x_{i,j} = 0$ if i losses to j
- There is a tournament with the given score sequence if and only if the following has a solution:

$$\sum_{i<j}(1 - x_{i,j}) + \sum_{j<k} x_{j,k} = s_k \text{ for } j = 1, 2, \ldots, n$$
$$x_{i,j} \in \{0, 1\} \text{ for all } i < j$$

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A certificate of inconsistency translates to a violation of Landau's necessary condition

In general solving $A\mathbf{x} = \mathbf{b}$ subject to the condition that the entries of $\mathbf{x}$ are $0, 1$ is an NP-hard problem. This implies in a certain sense that there is no theorem analogous to Farkas' Lemma for linear systems with $0, 1$ constraints.

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In fact Landau's system becomes a special case of Hoffman's circulation theorem

Fishburn's Theorem via systems of linear inequalities

- Consider variables r_v and l_v for the placement of the right and left endpoints of the intervals.
- A given digraph has an interval representation if and only if the following has a solution:

$r_v < l_w$ if v comes before w
$r_v \geq l_w$ if v does not come before w
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In fact this system is a special case of finding shortest paths in a digraph

Hoffman's Circulation Theorem via systems of linear inequalities

- A network with upper bounds $u(xy)$ and lower bounds $l(xy)$ for arcs xy has a feasible circulation with flows $f(xy)$ if and only if the following system has a solution:

$\sum_{xy \in A} f(xy) - \sum_{yz \in A} f(yz) = 0$ for all vertices $y \in V$ flow conservation constraints

$f(xy) \leq u(xy)$ upper bounds on flow

$l(xy) \leq f(xy)$ lower bounds on flow

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The simple question

When does

$$\begin{aligned} x &\leq a \\ x &\geq 0 \end{aligned}$$

have a solution?

leads to some interesting mathematics