Degree Matrices Realized by Edge-Colored Forests

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Abstract

Given a c-edge-colored graph G on n vertices, we define the degree matrix M as the $c \times n$ matrix whose entry d_{ij} is the degree of color i at vertex v_j. We show that the obvious necessary conditions for a $c \times n$ matrix to be the degree matrix of a c-edge-colored forest on n vertices are also sufficient.

It is well known that non-negative integers $d_1, d_2, \dots, d_n$ form a degree list of a forest on n vertices if and only if $\sum d_i$ is even and $\sum d_i \leq 2s - 2$ where s is the number of non-zero entries in $d_1, d_2, \dots, d_n$. We are interested in similar conditions for edge colored forests where we specify the number of incident edges of each color. Given an edge colored forest and any set of colors, the edges of those colors induce a forest. Thus the degree sums for these colors must satisfy the conditions for uncolored forests. We will show that this necessary condition is also sufficient.

The three color version of our problem is related to results in [1] and [2]. In those papers, two of the three colors each induce a forest (but not necessarily their union) and the third color is the complement of their union. For our results, if every entry is $0, 1$ or 2 then the forest would be a disjoint union of paths. If we further specify the lengths of these paths, conditions for a realization become more complicated. We will examine these in a forthcoming paper.

We start by formalizing our notation.

Definition 1. *Given a graph G, a c* edge coloring *is an assignment $c : E_G \to [c]$ from its edge set into the set $[c] = \{1, 2, \dots, c\}$ whose elements are called* colors*. For any c, such an assignment is called an* edge-coloring*. If G is assigned such a coloring, then G is called an* edge-colored graph*.*

Note that c is not necessarily a proper edge coloring, i.e. two adjacent edges may have the same color.

Definition 2. *Let G be a c-edge colored graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The* degree *of color i at vertex v_j, denoted d_{ij}, is the number of color i edges incident to vertex v_j. The* degree *of vertex v_j is the vector $(d_{1j}, d_{2j}, \ldots, d_{cj})$ and the* degree matrix *M is the $c \times n$ matrix with entries d_{ij}.*

In order to more formally state the necessary conditions noted above we need some more notation. When not otherwise specified we will assume that M is a $c \times n$ matrix with entries d_{ij}.

Definition 3. *The* support *of the row set I in a matrix M, denoted by $S_M(I)$, is the set of columns with non-zero entry in some row $i \in I$.*

Definition 4. *In a matrix M, let $T_M(I, J) = \sum_{i \in I} \sum_{j \in J} d_{ij}$ be the* total sum *of the entries in rows $i \in I$ and columns $j \in J$. We use $T_M(I)$ for $T_M(I, [n])$ for M with n columns when there is no chance of confusion.*

Lemma 5. *Necessary conditions for a given $c \times n$ matrix M to be the degree matrix of a c-edge-colored forest on n vertices are as follows:*

1. *All entries d_{ij} for $1 \leq i \leq c$, $1 \leq j \leq n$ must be non-negative integers.*
2. *Each row sum $\sum_{j=1}^{n} d_{ij}$ for $i = 1, 2, \ldots, c$ must be even.*
3. *$T_M(I) \leq 2|S_M(I)| - 2$ for all $I \subseteq [c]$.*

Condition (3) states that the sum of a subset of rows must satisfy the necessary conditions for an uncolored forest.

Throughout the remainder of this paper, we will refer to the three conditions above as "the obvious necessary conditions".

For completeness we include a proof of the observation of Lemma 5 that these are necessary conditions.

Proof. Let M be the degree matrix of a c edge colored forest F on n vertices. For condition (1), by Definition 3 we know d_{ij} is the number of color i edges at vertex v_j. Therefore d_{ij} is a non-negative integer. Each edge of color i contributes to the degree of two vertices so every row sum must be even as in condition (2). The subgraph of F induced by the edges of color $i \in I$ is also a forest. The number of vertices with non-zero degree in this forest is given by $|S_M(I)|$. Therefore $T_M(I) \leq 2|S_M(I)| - 2$ for all $I \subseteq [c]$ resulting in condition (3). $\square$

In Theorem 13 at the end of this paper, we show that these obvious necessary conditions are also sufficient. For our proof we consider two cases. If strict inequality holds in condition (3) for some proper subset of colors then we inductively construct a tree on the vertices with non-zero degree in these colors, collapse these columns to correspond to a single vertex, construct a forest for the remainder and patch the two parts together. If strict inequality holds in (3) for all subsets of colors then we remove a column with exactly one entry 1 and the rest 0, reduce one other entry, inductively construct a tree and add a leaf corresponding to the column that was removed. The construction for the second case could also be used in the first case but we must be careful in selecting the other entry to reduce to ensure that the conditions still hold. The two case approach that we use seems to be somewhat less complicated. The example after lemma 9 will further illustrate this.

We state several lemmas to establish that various reductions to matrices preserve the obvious necessary conditions.

The first lemma shows that we can eliminate a row of zeros as it would correspond to a color that is not used and we can eliminate a column of zeros as it would correspond to an isolated vertex.

Lemma 6. *If M satisfies the obvious necessary conditions to be the degree matrix of a c-edge-colored forest on n vertices and the matrix $\hat{M}$ is obtained from M by deleting a row of all zeros or a column of all zeros, then $\hat{M}$ satisfies the obvious necessary conditions to be the degree matrix of an edge-colored forest.*

Proof. Given an M that satisfies the obvious necessary conditions and has a column (or row) of all zeros, we delete this column (or row) to obtain $\hat{M}$. The necessary conditions (1) and (2) of Lemma 5 are not affected.

If we delete a row or column of zeros we have $T_M(I) = T_{\hat{M}}(I)$ and $|S_M(I)| = |S_{\hat{M}}(I)|$ for all subsets I of the rows of $\hat{M}$. Therefore (3) holds. $\square$

We now can make an observation that the conditions imply that a matrix satisfying the conditions has a column corresponding to a leaf in a forest realizing the matrix.

Lemma 7. *If M is nonzero and satisfies the obvious necessary conditions to be the degree matrix of a c-edge-colored forest on n vertices, then M will have some column with one element equal to 1 and the remaining elements equal to 0.*

Proof. By Lemma 6 we can assume M does not have a column of all zeros. Hence $S_M([c]) = n$. If there is no such column, $T_M([c], \{j\}) \geq 2$ for $j = 1, 2, \ldots, n$. Thus $T_M([c]) \geq 2n$, contradicting $T_M([c]) \leq 2S_M([c]) - 2 = 2n - 2$. $\square$

Permuting rows and columns does not affect the obvious necessary conditions. Thus, if M satisfies the obvious necessary conditions to be the degree matrix of a c-edge-colored forest on n vertices we can assume that the last column has first entry 1 and all other entries 0. Further, we can assume $d_{1(n-1)} \geq 1$ since each row sum must be even.

Construction 8. *Let M be a $c \times n$ matrix with no zero columns, $d_{1,(n-1)} \geq 1$, $d_{1n} = 1$ and for $i = 2, 3 \ldots, c$, $d_{in} = 0$. Form a new matrix M' by reducing the first entry in column $(n-1)$ by 1 and deleting the n^{th} column.*

In the next lemma note that we look at strict inequality for condition (3) in proper subsets of the rows.

Lemma 9. *If M has no zero rows or columns and satisfies the obvious necessary conditions to be the degree matrix of a c-edge-colored forest on n vertices and also $T_M(I) < 2|S_M(I)| - 2$ for all $I \subset [c]$, then there exists a matrix M' formed as in Construction 8 that satisfies the obvious necessary conditions to be the degree matrix of a c-edge-colored forest on $n-1$ vertices.*

Proof. Clearly such an M' satisfies the first two obvious necessary conditions. For some choice of M' we need to show condition (3), $T_{M'}(I) \leq 2|S_{M'}(I)| - 2$ for all $I \subseteq [c]$. We know $T_M(I) < 2|S_M(I)| - 2$ for all $I \subset [c]$ and by condition (2), $T_M(I)$ is even. Thus $T_M(I) \leq 2|S_M(I)| - 4$ for all $I \subset [c]$.

By Lemma 7 we can assume that column n has the form for Construction 8. If also $T_M([c]) = 2n - 2$ then we will also permute the columns so that column $n-1$ has at least one nonzero among rows $2, \ldots, c$. If we cannot do this then $S_M(\{1\}) = t$ and $S_M([c] - \{1\}) = n - t$ for some t. Since condition (3) is a strict inequality for these, $T_M(\{1\}) < 2t - 2$ and $T_M([c] - \{1\}) < 2(n-t) - 2$. Then $T_M([c]) < 2t - 2 + 2(n-t) - 2 = 2n - 4$, a contradiction. Thus we can pick column $n-1$ as stated when $T_M([c]) = 2n - 2$

We consider 3 cases: (i) row $1 \notin I$, (ii) row $1 \in I$, $I \neq [c]$ and (iii) $I = [c]$.

Case (i), row $1 \notin I$:
M and M' are identical on the rows in I except that M' has the last column of all zeros deleted. Thus by Lemma 6 applied to the submatrix consisting of rows of I, we see that condition (3) holds.

Case (ii), row $1 \in I$, $I \neq [c]$:
$|S_{M'}(I)| \geq |S_M(I)| - 2$. Then $T_{M'}(I) = T_M(I) - 2 \leq (2|S_M(I)| - 4) - 2 \leq (2(|S_{M'}(I)| + 2) - 4) - 2 = 2|S_{M'}(I)| - 2$. So condition (3) holds.

Case (iii), $I = [c]$:
If $T_M([c]) < 2n-2$, condition (3) holds as in case (ii). Otherwise, we have selected column $n-1$ so that $|S_{M'}([c])| = |S_M([c])| - 1$. Then $T_{M'}([c]) = T_M([c]) - 2 \leq (2|S_M([c])| - 2) - 2 = (2(|S_{M'}([c])| + 1) - 2) - 2 = 2|S_{M'}([c])| - 2$. So condition (3) holds. ☐

To illustrate that we must be careful in which column to pick to be column $n-1$ in Lemma 9, consider $\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$. Reducing the ones in the last two columns of row 1 produces M' that violates the necessary conditions. However, if we permute so that column three is the new column seven the construction of M' results in a matrix that still satisfies the necessary conditions. This also illustrates the complications in a proof for which we always use induction on a matrix like that from Construction 8. In selecting column $n-1$ to ensure that the necessary conditions hold in M' we either need $d_{1,(n-1)} \geq 2$ or we need a nonzero in column $n-1$ for some row of I for all I such that condition (3) holds with equality for $I \cup \{1\}$. Instead we use the following approach when we get equality in condition (3) for some strict subset of $[c]$.

We use $\bar{I}$ to denote the complement.

Construction 10. *Let M be a $c \times n$ matrix and $I \subset [c]$. Form the following two new matrices:*
$M_1(I)$ has rows from I and columns from $S_M(I)$.
$M_2(\bar{I})$ has rows from $\bar{I}$ and columns in $S_M(I \cup \bar{I}) - S_M(I)$ with an additional first column formed by summing, for each row in $\bar{I}$, the entries of the columns in $S_M(I)$.

We illustrate with matrix $M = \begin{pmatrix} M_1(I) & 0 \\ Q & R \end{pmatrix}$. The columns of M have been permuted such that the support of the row set I is listed first and the rows have been permuted such that rows in I are listed first. The matrix $M_2(I)$ is R with an additional (first) column with entries equal to the row sums in Q.

Lemma 11. *If M satisfies the obvious necessary conditions to be the degree matrix of a c-edge-colored forest on n vertices and $T_M(I) = 2|S_M(I)| - 2$ for some $I \subset [c]$, then the submatrix $M_1(I)$ will satisfy the obvious necessary conditions to be the degree matrix of an $|I|$-edge-colored forest on $|S_M(I)|$ vertices.*

Proof. Let M^* be the $(|I| \times n)$ matrix consisting of all the rows in I. Then M^* satisfies the obvious necessary conditions as deleting rows does not affect the conditions. Now

$M_1(I)$ is obtained from M^* by possibly deleting a column or columns of all zeros. By Lemma 6, $M_1(I)$ satisfies the necessary conditions. $\square$

Lemma 12. *If M satisfies the obvious necessary conditions to be the degree matrix of a c-edge-colored forest on n vertices and $T_M(I) = 2|S_M(I)| - 2$ for some $I \subset [c]$, then the submatrix $M_2(\bar{I})$ will satisfy the obvious necessary conditions to be the degree matrix of an $|\bar{I}|$-edge-colored forest on $n - |S_M(I)| + 1$ vertices.*

Proof. Clearly, $M_2(\bar{I})$ satisfies the first two necessary conditions. We write M_2 for $M_2(\bar{I})$. We need to show condition (3), $T_{M_2}(B) \leq 2|S_{M_2}(B)| - 2$ for all $B \subseteq \bar{I}$.

We consider two cases: (i) $T_M(B, S_M(I)) > 0$ and (ii) $T_M(B, S_M(I)) = 0$.

Case(i), $T_M(B, S_M(I)) > 0$:
In this case $|S_{M_2}(B)| = |S_M(I \cup B)| - |S_M(I)| + 1$. From the necessary conditions we have $T_M(I \cup B) \leq 2|S_M(I \cup B)| - 2$. From the construction and since $I \cap B = \emptyset$ we have $T_{M_2}(B) = T_M(B) = T_M(I \cup B) - T_M(I)$. From the choice of I, $T_M(I) = 2|S_M(I)| - 2$. Hence $T_{M_2}(B) = T_M(I \cup B) - T_M(I) \leq 2|S_M(I \cup B)| - 2 - 2|S_M(I)| + 2 = 2(|S_M(I \cup B)| - |S_M(I)| + 1) - 2 = 2|S_{M_2}(B)| - 2$. Thus condition (3) holds.

Case (ii), $T_M(B, S_M(I)) = 0$:
In this case $|S_{M_2}(B)| = |S_M(I \cup B)| - |S_M(I)|$. Similar to the proof of Lemma 11, we let M^{**} be the $(|B| \times n)$ submatrix of M consisting of all the rows in B. Then M^{**} satisfies the necessary conditions as it is a subset of rows of M. Now M_2 is obtained from M^{**} by deleting a columns or columns of all zeros. By Lemma 6, condition (3) holds. $\square$

With these lemmas, we are now able to prove our main theorem.

Theorem 13. *Given a $c \times n$ matrix M, the obvious necessary conditions established in Lemma 5 are sufficient for M to be the degree matrix of a c-edge-colored forest on n vertices.*

Proof. We use induction on the number of rows plus the number of columns of M. Note that M will always have at least two columns except the trivial case of a column of all zeros. When M has only one row, the conditions are the well known conditions for a sequence to be forest realizable. Therefore, we consider matrices of two or more rows and columns.

We consider three cases: (i) M has a row or column of all zeros, (ii) $T_M(I) < 2|S_M(I)| - 2$ for all $I \subset [c]$ and (iii) $T_M(I) = 2|S_M(I)| - 2$ for some $I \subset [c]$.

Case (i), M has a row or column of all zeros:
By Lemma 6 deleting such a row or column does not affect the conditions. By

induction we form an edge colored forest and add isolated vertices for each column that was deleted. Deleted rows correspond to unused colors.

Case (ii), $T_M(I) < 2|S_M(I)| - 2$ for all $I \subset [c]$:
By Lemma 9 we can construct a matrix M', using Construction 8, that satisfies the obvious necessary conditions. Assume M' has rows labeled $1, 2, \ldots, c$ and columns labeled $1, 2, \ldots, n-1$. The number of columns in M' is smaller than in M. By induction, there exists a forest F' with vertices $1, \ldots, n-1$ and edges colored 1 through c that has degree matrix M'.

We add a new vertex v_n to this forest F' and attach an edge $v_n v_{n-1}$ of color 1. We call this new graph F and see that in F, the number of color 1 edges incident to v_{n-1} is now $d_{1(n-1)} - 1 + 1 = d_{1(n-1)}$. Hence, the result is a new forest F on n vertices that has degree matrix M. We note that F is a forest since adding a leaf to F' does not create a cycle.

Case (iii), $T_M(I) = 2|S_M(I)| - 2$ for some $I \subset [c]$:
By Lemmas 11 and 12 we can construct matrices $M_1(I)$ and $M_2(\bar{I})$, using Construction 10, that satisfy the obvious necessary conditions. Assume $M_1(I)$ has rows labeled $1, 2, \ldots, a-1$ and columns labeled $1, 2, \ldots, k$. Also assume $M_2(\bar{I})$ has rows labeled $a, a+1, \ldots, c$ and columns labeled $0, k+1, \ldots, n$. Since $I \subset [c]$ and $S_M(I) \geq 2$, the number of rows plus columns in M_1 and in M_2 is smaller than in M.

By induction, there exists a forest F_1 with vertices $1, \ldots, k$ and edges colored 1 through $a-1$ that has degree matrix $M_1(I)$. The vertex set of the forest F_1 is given by $V(F_1) = \{v_1, v_2, \ldots, v_k\}$ and the edge set of the forest F_1 is given by $E(F_1) = E_1(F_1) \cup E_2(F_1) \cup \ldots \cup E_{(a-1)}(F_1)$ where $E_x(F_1)$ denotes the edges of color x in F_1 for $x \in \{1, 2, \ldots, (a-1)\}$.

Similarly, there exists a forest F_2 with vertices $0, k+1, \ldots, n$ and edges colored a through c that has degree matrix $M_2(\bar{I})$. The vertex set of the forest F_2 is given by $V(F_2) = \{v_0, v_{k+1}, \ldots, v_n\}$ and the edge set of the forest F_2 is given by $E(F_2) = (E'_a(F_2) \cup \cdots \cup E'_c(F_2)) \cup (E''_a(F_2) \cup \cdots \cup E''_c(F_2))$ where $E'_x(F_2)$ denotes the edges of color x not incident to v_0 and $E''_x(F_2)$ denotes the edges of color x incident to v_0.

Considering $E''_x(F_2)$, we let $W_x = \{v_m | v_0 v_m \in E''_x(F_2)\}$ be the set of all vertices that have a color x edge to v_0. From the construction of $M_2(\bar{I})$, $|W_x| = \sum_{i=1}^{k} d_{xi}$. Take any partition $W_x = W_{x1} \cup W_{x2} \cup \cdots \cup W_{xk}$ with $|W_{xi}| = d_{xi}$.

We now use F_1 and F_2 to construct the forest F with vertex set given by $V(F) = \{v_1, v_2, \ldots, v_n\}$ and edge set given by $E(F) = E_1(F) \cup E_2(F) \cup \cdots \cup E_c(F)$ where

$$E_x(F) = \begin{cases} E_x(F_1) \text{ if } x \in \{1, 2, \ldots, a-1\} \\ E'_x(F_2) \cup \{v_i v_m | i = 1, 2, \ldots, k \text{ and } m \in W_{xi}\} \text{ if } x \in \{a, a+1, \ldots, c\} \end{cases}.$$

F_2 can be obtained by contracting the vertices of F_1 in F. Any cycle in F would either be contained in F_1 or would become a cycle in F_2. Since F_1 and F_2 have no cycles, F does not contain a cycle. So F is a forest. It is straightforward to check that F has degree matrix M. $\square$

References

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