Functions

1. A company sells 100 widgets at a price of \$20. Sales increase by 5 widgets for each \$1 decrease in price. Write an expression for the number q of widgets sold in terms of the price p.

   (a) Implicit in the description is that this is a linear relationship. The information tells us that it goes through the point $(p, q) = (20, 100)$, the quantity q sold is 100 when the price p is \$20. Also we see that at \$19 sales are 105.

   (b) Recalling some method for finding an equation for a line through two points you will find that the line is $q = -5p + 200$. We will review equations of lines in Lecture 02.

   (c) Note that the slope of the line -5 corresponds to the description that an increase in price of \$1 decreases sales by 5.

   In a model like this we should also limit the domain. It would not make sense to consider negative price, for example. The reaction to price changes would only be valid over a small ranges of prices around \$20 and the function should only be used in that range.

2. Example 1 describes a function with price as input and quantity as output. It is called the *demand curve (or demand function)*. For each input there is a unique output. We can evaluate the function for different values of price. It is sometimes convenient to write $q(p) = -5p + 200$. Here we use the symbol p to represent the input and call this the independent variable. Here we use q for the output and call it the dependent variable.

   Then $q(15) = -5(15) + 200 = 125$. We have just evaluated the function at 15 with interpretation that sales are 125 when price is \$15.

   (a) The function $f(x) = -5x + 200$ is the same as $q = -5p + 200$. We just used different symbols for the variables. We also sometimes write $y = -5x + 200$.

   (b) We could write $\clubsuit(\diamondsuit) = -5\diamondsuit + 200$ or $brown(white) = -5 \cdot white + 200$ although typically we do not use such symbols. The symbols just represent the variables. They have no special meaning. We just pick something convenient to work with and possibly remind us of what they represent in an application.

   (c) We will also use other symbols that represent constants. For example, working with interest rates it is convenient to use r rather than, say .04 because this allows understanding of general situations beyond a rate of 4%.

   In any problem it is useful to identify what each symbol represents. It might be a function, an independent variable, a dependent variable or a constant.

3. A *function* is a rule that assigns each element in a set D, the *domain*, a unique value in a set R, the *range*. For example, at the end of class I will define a function that assigns a grade to each student. A symbol representing an arbitrary element of the domain is an (*independent*) variable (also called the argument) and a symbol representing an arbitrary element of the range is a (*dependent*) variable (also called the value). For the function above, when we wrote $y = -5x + 200$, x is the independent variable and y the the dependent variable. When we wrote $q = -5p + 200$, p is the independent variable and q the the dependent variable.

   (a) Usually we will work with functions where the domain and range are real numbers and the function is specified algebraically.

   (b) Later we will consider functions with several independent variables. In the example above, If we wanted a function to represent the cost of manufacturing widgets, it might depend on costs of several different types of raw materials as well as labor costs etc. each of which would be represented by an independent variable. (Technically the domain in such a case is a set of lists of several numbers.)

4. The height in meters, after t seconds, of an object thrown upwards with a velocity of 15 meters per second is $h(t) = -4.9t^2 + 15t$. We can evaluate the function by substituting a particular value for t. For example, after 2 seconds the height is $h(2) = -4.9(2^2) + 15(2) = 10.4$ meters.

   (a) This assumes gravity on the surface of the earth and no air resistance. The units are meters and seconds. In this class we will not focus on the units of the variables. In applications it is important to keep track of appropriate units.

   (b) The function above applies to one particular situation. It is more useful to consider general models that apply in other settings. For example, at high altitudes the force due to gravity is slightly different and it is very different on the moon or Mars.

   (c) Instead of numbers we want to represent a generic force due to gravity and a generic initial velocity. We have $h(t) = \frac{G}{2}t^2 + v_0 t$ where G represents acceleration due to gravity and v_0 (read v naught) represents initial velocity. Here we have a function with independent variable t and several constants G and v_0. In a particular application of the model G and v_0 will be different. When we work with such functions we will treat these constants as if they are some number specified by a particular symbol.

   (d) One problem we will be interested in is how high the object goes. With some easy calculus we will be able to solve this.

(e) Historically calculus was developed independently by Isaac Newton in England and Gottfried Leibniz in Germany in the late 1600's. Newton developed calculus to understand planetary motion.

5. The function $A(t) = 1000(1 + \frac{.03}{12})^{12t}$ gives the amount in an account starting with \$1000, after t years if annual interest rate is 3%, compounded monthly. Principal, interest and compounding period are not always the same so it's convenient to use constants. For t years, principal P and annual interest rate r (expressed as a decimal) compounded n times per year we get $A(t) = P(1 + \frac{r}{n})^{nt}$.

   (a) Depending on situation we might be interested in looking at what happens to the amount as principal, time, compounding period or rate changes. So we could, in different situations consider any of P, t, n, r to be the independent variable and the others constants. Alternatively we could consider it a function of four independent variables if we are interested in how different changes interact.

   (b) As the number of periods n increases we should get better and better approximations to continuous compounding. Compounding every second would have $n = 31,536,000$. For continuous compounding we want to understand what happens as n gets very large. We will call this the limit as n approaches infinity. It turns out that taking this limit gives $B(t) = Pe^{rt}$ where $e = 2.718\ldots$ is a particular irrational number called Euler's number. That is, $(1 + \frac{r}{n})^n$ gets closer and closer to e^r as n gets larger.

6. What happens in example 1 if you need to pay a \$3 bribe to the local cartel or if payment is in yuan instead of dollars? These are examples of transformations of functions. The graph $q = -5p + 200$ is a line.

   (a) If we want to determine sales for a given amount of income received per item we need to shift the line by 3 to account for the bribe. For example, to receive \$17 you must charge the customer \$20 to cover the bribe. So for the new function sales at \$17 are the same as sales at \$20 in the original. The new function is the old shifted 3 units to the left. The new function is $q(p+3) = -5(p+3) + 200 = -5p + 185$.

   (b) Measured in yuan, the input is scaled by the conversion between yuan and dollars. Currently it is (about) 6.5 yuan per dollar. Thus if we want input in yuan instead of dollars we stretch the horizontal axis by a factor of 6.5 so that a new input of 6.5 corresponds to an old input of 1. The new function is $h(\frac{p}{6.5}) = -5(\frac{p}{6.5}) + 200 = \frac{-5}{6.5}p + 200$.

   (c) Text pages 17-18 illustrates various graph transformations. You should know these.

(d) Note that horizontal effects are represented in the input to the function and vertical effects are changes to the function.

Note also the somewhat counterintuitive form for horizontal effects. $f(x+5)$ is the graph of $f(x)$ shifted 5 units left. One way to remember this is to think about what value makes $x+5=0$. It is -5. So in the shifted graph the input -5 has the same output (height) as input 0 has in the original. We have shifted left 5 units.

$f(7x)$ is the graph of $f(x)$ scaled by a factor of $\frac{1}{7}$. Here think about what value makes $7x=1$. It is $\frac{1}{7}$. So in the scaled graph input $\frac{1}{7}$ has the same output (height) as input 1 in the oriignal. We have scaled by a factor of $\frac{1}{7}$, i.e., we have compressed the graph horizontally by a factor of 7. $f(\frac{1}{3}x)$ is scaled by a factor of 3, i.e., it is stretched by a factor of 3.

Note that we will use the terms compressed and stretched rather than broadened and steepened in the text.

(e) The graph of $y=x^2$ is a parabola with vertex at the origin. The graph of $y=3(x-7)^2+13$ takes the basic parabola, stretches the vertical axis by 3, shifts the graph right by 7 and the shifts it up by 13.

7. Basic functions that we will use include *algebraic functions*, those built using basic algebraic operations and three classes (which we will explore first in lectures 3 and 4) of *transcendental functions*: exponentials, logarithms and trigonometric functions. Text page 10.

(a) Examples of general algebraic functions include $f(x)=(2x-7)^{-2/3}+\frac{x+1}{x-2}$ and $E=(mc^2)/\sqrt{1-v^2/c^2}$ which is Einstein's formula for energy in terms of velocity v, mass m and speed of light c and $Q=cK^{\alpha}L^{\beta}$ which is the Cobb-Douglas function for economic output in terms of capital K, labor L with constants α, β and c depending on the system.

(b) Examples of *polynomial functions* (a special class of algebraic functions) are $f(x)=13x^5-7x^2+42$ and $q(p)=3p^2-p_2$. Appropriate notation allows representing a general polynomial as $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ where n is a nonnegative integer and the a_i are constants with $a_n\neq 0$. Text page 13.

(c) Examples of *rational functions* (a special class of algebraic functions that includes polynomials) are $f(x)=\frac{3x^2-5x+7}{42x^5+19}$. In general a rational function is $f(x)=\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. Text page 14.

8. Once we introduce constants we can evaluate functions at constants as well as specific numbers. The symbolic manipulation in the sort of function evaluation we now consider is to simply replace each instance of the independent variable with a new symbol and simplify if possible.

   (a) With a general $f(x)$ for example, we find $f(2)$ by replacing each x with a 2 and then carrying out the arithmetic. Similarly, we find $f(\gamma+7)$ by replacing each x with $\gamma+7$. We may then simplify the expression if it is convenient.
   If $f(x) = x^2 + x + 1$ then
   $f(0) = 0^2 + 0 + 1 = 1$
   $f(2) = 2^2 + 2 + 1 = 7$
   $f(a) = a^2 + a + 1$
   $f(\gamma+7) = (\gamma+7)^2 + (\gamma+7) + 1 = (\gamma^2 + 14\gamma + 49) + (\gamma+7) + 1 = \gamma^2 + 15\gamma + 57.$
   For $f(a)$ there was nothing to do to simplify. We will review the algebra for expanding quadratics in Lecture 03. Sometimes there is not much that can be done to simplify.

   (b) If $g(t) = s^3t^2 - \frac{t}{s}$ then find $g(3), g(r+5)$ and $g(s^2)$. Here, since t is the independent variable we replace each instance of t as indicated and do nothing with the $s's$.
   $g(3) = s^3 \cdot 3^2 - \frac{3}{s} = 9s^3 - \frac{3}{s}$
   $g(r+5) = s^3(r+5)^2 - \frac{r+5}{s}$
   $g(s^2) = s^3 \cdot (s^2)^2 - \frac{s^2}{s} = s^3 \cdot s^4 - s = s^7 - s.$
   We could expand in the second example but it will not make things much simpler. Note that the $s's$ arise both from the function and the substitution in the third example. We will review rules of exponents as in the third example in Lecture 02.

9. We just saw examples where we evaluate a function at a new function value. For example, $h(x) = \sqrt{x^2+1}$ is $f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$ where $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$. This is an example of a *compound function*. Breaking down functions this way into simpler pieces will be useful. Other notation is $(f \circ g)(x) = f(g(x))$. Text page 3.

   Symbolically we are just evaluating the function but the notation can be a bit confusing since in the evaluation we replace each instance of x with another expression involving x. So x has a dual role.

Supplementary problems

Evaluating functions

P1.1 If $R(Q) = KQ^2 - P + \gamma\frac{P}{Q}$ find $R(7)$ and $R(P^3)$ and $R(K)$.

P1.2 If $f(x) = \alpha x^2 - \frac{\beta^2}{x}$ find $f(3)$ and $f(x+h)$.

P1.3 If $f(x) = x^2 - x + 2$ find $f(-2)$, $f(a^2)$ and $[f(a)]^2$

Transforming functions

P1.4 Describe how to obtain $f(x) = (x-4)^2 + 17$ and $g(x) = (2x-4) + 17$ using transformations of $h(x) = x^2$.

P1.5 If $h(x) = \sqrt{x}$ give a formula for $f(x)$ obtained by shifting $h(x)$ left by 3 units and down by 5 units, and $g(x)$ obtained by stretching $h(x)$ horizontally by a factor of 7 and vertically by a factor of 9.

Composite functions

P1.6 If $f(x) = x^2 + 7$ and $g(x) = \sqrt{x}$ find $f(g(x))$ and $g(f(x))$.

P1.7 If $U(p) = p - \frac{1}{p}$ and $V(p) = 3p + 7$ find $U(V(p))$ and $V(U(p))$.

P1.8 If $h(x) = (ax^2 + bx + c)^7$ write $h(x) = f(g(x))$ for appropriately chosen $f(x)$ and $g(x)$.

P1.9 If $h(x) = 2^{x^2+x+1}$, write $h(x) = f(g(x))$ for appropriately chosen $f(x)$ and $g(x)$.

Solutions to supplementary problems

Evaluating functions

S1.1 $R(7) = K\,7^2 - P + \gamma\frac{P}{7} = 49K - P + \frac{\gamma P}{7}$.
$R(P^3) = K(P^3)^2 - P + \gamma\frac{P}{P^3} = KP^6 - P + \frac{\gamma}{P^2}$.
$R(K) = K \cdot K^2 - P + \gamma\frac{P}{K} = K^3 - P + \gamma\frac{P}{K}$

S1.2 $f(3) = \alpha\,3^2 - \frac{\beta^2}{3} = 9\alpha - \frac{\beta^2}{3}$.
$f(x+h) = \alpha(x+h)^2 - \frac{\beta^2}{x+h}$.

S1.3 $f(-2) = (-2)^2 - (-2) + 2 = 8$
$f(a^2) = (a^2)^2 - a^2 + 2 = a^4 - a^2 + 2$
$[f(a)]^2 = (a^2 - a + 2)^2 = a^4 - 2a^3 + 5a^2 - 4a + 4$

Transforming functions

S1.4 $f(x) = (x-4)^2 + 17$ is obtained by shifting $h(x) = x^2$ right 4 units and up 17 units.
$f(x) = (2x-4)^2 + 17$ is obtained by shifting $h(x) = x^2$ by compressing by a factor of 2 then shifting right by 2 new horizontal units then shifting up by 17 units.
An alternative description is that it is obtained by shifting right by 4 units then compressing by 2 units and shifting up by 17 units.

S1.5 $f(x) = \sqrt{x+3} - 5$
$g(x) = 9\sqrt{\frac{1}{7}x}$

Composite functions

S1.6 $f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 7 = x + 7$
$g(f(x)) = g(x^2 + 7) = \sqrt{x^2 + 7}$

S1.7 $U(V(p)) = U(3p+7) = (3p+7) - \frac{1}{3p+7}$
$V(U(p)) = V(p - \frac{1}{p}) = 3(p - \frac{1}{p}) + 7 = 3p - \frac{3}{p} + 7$

S1.8 $h(x) = f(g(x))$ for $f(x) = x^7$ and $g(x) = ax^2 + bx + c$

S1.9 If $f(x) = 2^x$ and $g(x) = x^2 + x + 1$ then $f(g(x)) = f(x^2 + x + 1) = 2^{x^2+x+1}$