New Constructions for De Bruijn Tori

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Dedicated to the memory of Tony Brewster

Abstract

A De Bruijn torus is a periodic $d-$dimensional $k-$ary array such that each $n_1 \times \cdots \times n_d$ $k-$ary array appears exactly once with the same period. We describe two new methods of constructing such arrays. The first is a type of product that constructs a k_1k_2-ary torus from a k_1-ary torus and a k_2-ary torus. The second uses a decomposition of a d-dimensional torus to produce a $d+1$ dimensional torus. Both constructions will produce two dimensional $k-$ary tori for which the period is not a power of k. In particular, for $k = \Pi p_l^{\alpha_l}$ and for all natural numbers (n_1, n_2), we construct 2-dimensional $k-$ary De Bruijn tori with order $\langle n_1, n_2\rangle$ and period $\langle q, k^{n_1 n_2}/q\rangle$ where $q = k\Pi p_l^{\lfloor \log_{p_l} n_1 \rfloor}$.

1 Introduction

A *$d-$dimensional De Bruijn torus* B with *base* k, *order* $\vec{N} = \langle n_1, n_2, \ldots, n_d\rangle$ and *period* $\vec{R} = \langle r_1, r_2, \ldots, r_d\rangle$ is an infinite periodic array with period $\vec{R}$ and entries from $[k] = \{0, 1, \ldots, k-1\}$ ('*k-ary*') such that every k-ary matrix of size $\vec{N}$ appears exactly once periodically with period $\vec{R}$. We call such an array a $(\vec{R}; \vec{N})^d_k$ *De Bruijn torus* and denote the set of all such arrays by $dB^d_k(\vec{R}; \vec{N})$.

A *fundamental block* of B is an array consisting of r_i consecutive rows in the i^{th} dimension for $i = 1, 2, \ldots, d$. Repeating such a block produces B. We will sometimes refer to a fundamental block of B as B when there is no chance of confusion. Thus, we will say that a matrix appears uniquely in an infinite periodic array if it appears uniquely in a fundamental block. In this case addition on subscripts in the i^{th} dimension is performed modulo r_i and we think of B toroidally. Both perspectives,

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viewing B as a toroidal array and viewing B as an infinite periodic array have been used in the study of De Bruijn tori. We use the infinite array version here to simplify notation.

We will say that a particular matrix M of size $\vec{N}$ *appears in* B *at position* $\vec{I} = (i_1, i_2, \ldots, i_d)$ if M appears in the positions $\vec{I}$ through $\vec{I} + \vec{N}$. If $B = [b_{\vec{I}}]$ is any d-dimensional torus, the *projection of* B *along* $i_j = h$ is the $d-1$-dimensional torus consisting of all entries $a_{\vec{I}}$ for which $i_j = h$. So, for example, in 2 dimensions the projection along $i_2 = 7$ is the 7^{th} column (after the 0^{th} column).

A 1-dimensional De Bruijn torus is what has come to be known as a *De Bruijn cycle*. (See Fredricksen [4] for a survey of De Bruijn cycles.) Two dimensional De Bruijn tori are examined in Cock [1], Fan et al. [3] and others. See Hurlbert and Isaak [5] for references. A 2-dimensional De Bruijn torus $(r_1, r_2; n_1, n_2)^2_k$ is *square* if $n_1 = n_2$ and $r_1 = r_2$. Except for small values of n_j, it has been shown that the obvious necessary conditions $r_j > n_j$ and $r_1 r_2 = k^{n_1 n_2}$ are also sufficient for the existence of square tori. (See Fan et al. [3] when $k = 2$ and Hurlbert and Isaak [5] for the general case.) For non-square two-dimensional tori, Paterson has recently shown that the necessary conditions are also sufficient when the base k is a prime power. His methods include (among others) techniques like the those discussed in sections 2 and 3. See [8,9,10,11] for more details on these results.

For general De Bruijn tori, we believe that when the r_j are powers of the base k, the necessary conditions $\Pi r_j = k^{\Pi n_j}$ and $r_j > n_j$ are also sufficient. The constructions in this paper are a step towards resolving this question. The term product construction in Section 2 shows that it is sometimes enough to consider only the case in which k is a prime power. For example, we can construct a $(15^2, 15^4; 3, 2)_{15}$ torus from a $(3^2, 3^4; 3, 2)_3$ torus and a $(5^2, 5^4; 3, 2)_5$ torus. However, we cannot form a $(15^1, 15^5; 3, 2)_{15}$ torus from a $(3^1, 3^5; 3, 2)_3$ torus and a $(5^1, 5^5; 3, 2)_5$ torus because the base 3 torus does not exist. In particular, if k has prime factorization $k = \Pi p_i^{\alpha_1}$ then considering base $p_i^{\alpha_i}$ tori will suffice if for each $r_j = \Pi p_i^{\beta_{i,j}}$, we have $p_i^{\beta_{i,j}} > n_j$.

In Section 3, we develop a general method of constructing De Bruijn tori from decompositions of lower dimensional tori. In Section 4, this construction is applied to a particular decomposition of 1-dimensional De Bruijn cycles to prove our main result.

Theorem 1.1. *For all natural numbers* n_1, n_2, k *there exists a* $(q, k^{n_1 n_2}/q; n_1, n_2)^2_k$ *De Bruijn torus, where* k *has prime decomposition* $k = \Pi p_l^{\alpha_l}$ *and* $q = k \Pi p_l^{\lfloor \log_{p_l} n_1 \rfloor}$.

Observe that in most cases these periods will not be powers of k. This is the first construction of a general family for which the periods are not powers of k. (See Hurlbert and Isaak [6] for the case $n_1 = n_2 = 2$).

2 Term Products

Definition 2.1. *Let $B = [b_{\vec{I}}]$ be an infinite k-ary matrix and let $B' = [b'_{\vec{I}}]$ be an infinite k'-ary matrix. The term product $B \odot B'$ is the kk'-ary matrix with entry $\vec{I}$ given by $k'b_{\vec{I}} + b'_{\vec{I}}$.*

We will often find the notation $[B]_{\vec{I}}$ useful in place of $b_{\vec{I}}$, in which case the above definition may read $[B \odot B']_{\vec{I}} = k'[B]_{\vec{I}} + [B']_{\vec{I}}$. Note that the term product of B and B' could be viewed as having entries specified by the ordered pairs $(b_{\vec{I}}, b'_{\vec{I}})$. We have used a particular bijection between $[k] \times [k']$ and $[kk']$.

For example,

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{matrix}$$

is a fundamental block of a $(3,3;1,2)^2_3$ De Bruijn array B and $(0,0,1,1)$ is a fundamental block of a $(1,4;1,2)^2_2$ De Bruijn array B'. The term product $B \odot B'$ has period $\langle 3, 12 \rangle$. The ordered pairs giving rise to a fundamental region of the term product

$$\begin{matrix} (0,0) & (0,0) & (1,1) & (0,1) & (0,0) & (1,0) & (0,1) & (0,1) & (1,0) & (0,0) & (0,1) & (1,1) \\ (1,0) & (1,0) & (2,1) & (1,1) & (1,0) & (2,0) & (1,1) & (1,1) & (2,0) & (1,0) & (1,1) & (2,1) \\ (2,0) & (2,0) & (0,1) & (2,1) & (2,0) & (0,0) & (2,1) & (2,1) & (0,0) & (2,0) & (2,1) & (0,1). \end{matrix}$$

One can check that this term product is a $(3,12;1,2)^2_6$ De Bruijn array. As the following shows, this is no accident.

Theorem 2.2. *Let $\vec{R} = \langle r_1, \ldots, r_d \rangle$ and $\vec{R}' = \langle r'_1, \ldots, r'_d \rangle$ have $\gcd(r_j, r'_j) = 1$ for $j = 1, 2, \ldots, d$. If $B \in dB^d_k(\vec{R}; \vec{N})$ and $B' \in dB^d_{k'}(\vec{R}'; \vec{N})$ then $(B \odot B') \in dB^d_{kk'}(\vec{R}''; \vec{N})$ with $\vec{R}'' = \langle r_1 r'_1, r_2 r'_2, \ldots, r_d r'_d \rangle$.*

Proof. By the above remarks, we will view the entries of $B \odot B'$ as ordered pairs from $[k] \times [k']$. $B \odot B'$ is clearly periodic since it is periodic in both terms of the ordered pair. The period in the j^{th} dimension is the least common multiple of r_j and r'_j. This is $r_j r'_j$ by relative primality. Thus $\vec{R}''$ is the period.

Let S be any matrix of order $\vec{N}$ with entries from $[k] \times [k']$. Let S_1 be the order $\vec{N}$ matrix of first coordinates in S and S_2 the order $\vec{N}$ matrix of second coordinates. S_1 appears periodically in some location $\vec{I}$ in B and S_2 appears periodically in some location $\vec{I}'$ in B'. By the relative primality condition there is exactly one $\vec{0} \leq \vec{L} < \vec{R}''$ such that there exist $\vec{M} = \langle m_1, \ldots, m_d \rangle, \vec{M}' = \langle m'_1, \ldots, m'_d \rangle$ with $m_j r_j + [I]_j = m'_j r'_j + [I']_j = [L]_j$ for $j = 1, 2, \ldots, d$. At $\vec{L}$, S_1 appears in the first coordinate and S_2 appears in the second coordinate. Thus $\vec{L}$ is the unique location in the fundamental block located at $\vec{0}$ of $B \odot B'$ where S appears. $\square$

Corollary 2.3. *Let k and k' be relatively prime. If $B \in dB_k^d(\vec{R};\vec{N})$ and $B \in dB_{k'}^d(\vec{R}';\vec{N})$ then $B \odot B' \in dB_{kk'}^d(\vec{R}'';\vec{N})$ with $\vec{R}'' = \langle r_1 r_1', r_2 r_2', \ldots, r_d r_d' \rangle$.*

Proof. Since $k^{\Pi n_j} = \Pi r_j$, the factors of the r_j must be factors of k. Similarly, the factors of the r_j' must be factors of k'. If $\gcd(k,k') = 1$ then $\gcd(r_j, r_j') = 1$ for $j = 1, 2, \ldots, d$. □

3 Decomposition Construction

Ma [7] (in the binary case) and Cock [1] used De Bruijn cycles as building blocks for two-dimensional De Bruijn tori. Cock also notes (without proof) that the same construction works in higher dimensions. A similar technique is used by Etzion [2] to construct two-dimensional binary tori building on decompositions of De Bruijn tori. In this section we extend these techniques to construct $(d+1)$ dimensional De Bruijn tori from particular decompositions of d-dimensional tori. Our construction will in certain cases produce De Bruijn tori for which the entries in the period are not powers of the base k.

Definition 3.1. *A family $\mathcal{F} = \{F_0, \ldots, F_{t-1}\}$ of d-dimensional periodic k-ary arrays with period $\vec{R}$ such that each k-ary matrix of size $\vec{N}$ appears in exactly one of the arrays and it appears uniquely in that array is called a $d-$dimensional k-ary De Bruijn family with period $\vec{R}$, order $\vec{N}$ and size t.*

For example the strings $F_0 = (0,0,0,1)$ and $F_1 = (1,1,1,0)$ are fundamental blocks of a 1-dimensional binary De Bruijn family with period $\langle 4 \rangle$, order $\langle 3 \rangle$ and size 2. Each binary string of length 3 appears exactly once in one of the strings (viewed cyclically).

Lemma 3.2. *Let $\mathcal{F} = \{F_0, \ldots, F_{t-1}\}$ be a $k-$ary $d-$dimensional De Bruijn family of order $\vec{N} = \langle n_1, n_2, \ldots n_d \rangle$ and period $\vec{R} = \langle r_1, r_2, \ldots r_d \rangle$ with $n = \Pi n_j$ and $r = \Pi r_j$. Then $t = k^n / r$.*

Proof. There are r distinct matrices of size $\vec{N}$ in each F_l. There are k^n distinct $k-$ary matrices of size $\vec{N}$. Since each appears in exactly one of the arrays in the family, $tr = k^n$. □

Define the shift of a periodic sequence $A = (a_0, a_1, \ldots)$ to be the sequence $A^i = (a_i, a_{i+1}, \ldots)$. Construct a 2-dimensional array from the family $F_0 = (0,0,0,1)$ and $F_1 = (1,1,1,0)$ as follows: for each column, the entry of the binary string W determines whether the column is a cyclic shift of F_0 or F_1 and the entry of V determines the amount of shift. We get the following $(4, 16; 3, 2)^2_2$ De bruijn torus.

$V=$	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
$W=$	0	0	0	0	0	1	1	0	1	1	0	1	1	0	1	1
	0	0	0	1	0	1	0	0	1	1	0	0	1	0	0	1
	0	0	0	0	1	0	1	0	1	1	0	1	0	1	1	1
	0	0	1	0	0	1	1	1	1	1	1	1	1	0	1	0
	1	1	0	0	0	1	1	0	0	0	0	1	1	0	1	1

For example, with $[W]_7 = 0$, $[V]_7 = 3$, and $[W]_8 = 1$, the seventh column of the array being F_0^1 implies that the eighth column of the array is F_1^0. In general, if column j is $F_{[W]_j}^{s_j}$ then column $j+1$ is $F_{[W]_{j+1}}^{s_{j+1}}$, where $s_{j+1} = s_j + [V]_j$. For every binary pair (w, w') and 4−ary singleton (v) there is a unique j such that $(w, w') = ([W]_j, [W]_{j+1})$ and $(v) = ([V]_j)$. For example the pair $(0, 1)$ appears with (3) starting in the 7^{th} column. (The proof in the lemma below describes how to construct such pairs of strings.)

To find a particular 3 by 2 array, for example $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{\mathrm{T}}$, observe that $(0, 1, 0)$ appears as the first three digits of F_0^2 and that $(1, 1, 0)$ appears as the first three digits of F_1^1. The difference in these shifts is $1 - 2 \equiv 3 \pmod 4$. As noted above, $(0, 1)$ and (3) appear together starting in the seventh column and it is here that we find the array.

Theorem 3.3. *Let $\mathcal{F} = \{F_0, \ldots, F_{t-1}\}$ be a k−ary d−dimensional De Bruijn family of order $\vec{N} = \langle n_1, n_2, \ldots n_d\rangle$ and period $\vec{R} = \langle r_1, r_2, \ldots r_d\rangle$ with $r = \Pi r_j$. For any positive integer η, let $\beta = t^\eta$ and $\delta = r^{\eta-1}$. If any of the following hold:*
(i) $\gcd(\beta, \delta) = 1$ *or*
(ii) $\gcd(\beta, \delta - 1) = 1$ *or*
(iii) $\gcd(\beta - 1, \delta) = 1$
then (except when exactly one r_j is even and $\eta = 2$) there exists an $(\vec{R}'; \vec{N}')^{d+1}_k$ De Bruijn torus with $\vec{N}' = \langle n_1, n_2, \ldots, n_d, \eta\rangle$ and $\vec{R}' = \langle r_1, r_2, \ldots, r_d, r_{d+1}\rangle$, where $r_{d+1} = \beta\delta$.

Proof. We will give a construction of a $(d+1)$−dimensional array B to show the existence of the De Bruijn torus. If $\eta = 1$ simply let the projection of $B_{\vec{I}}$ along $i_{d+1} = h$ be F_h. Assume that $\eta \geq 2$. Let g be any bijection from $[\delta]$ to $[r_1] \times \cdots \times [r_d]$.

Assume first that $\gcd(\beta, \delta) = 1$. Let V be a t-ary De Bruijn cycle of order $\langle\eta\rangle$ and period $\langle\beta\rangle$. Let U be an r-ary De Bruijn cycle of order $\langle\eta - 1\rangle$ and period $\langle\delta\rangle$. Construct a new array B with the $\vec{I}^{th}$ entry given by

$$[B]_{\vec{I}} = F_{[V]_h}^{G(h)},$$

with $h = i_{d+1}$ and $G(h) = \sum_{j=0}^{h} g([U]_j)$.

To see that a De Bruijn array is created, note that since the periods of the F_l are the same in the first d dimensions, they remain so in B. For the $(d+1)^{st}$ dimension, note that the sequence of pairs $([V]_i, g([U]_i))$ is periodic with period $\langle \beta\delta \rangle$ since $\gcd(\beta, \delta) = 1$. We also need the j^{th} coordinate of $G(\beta\delta - 1)$ to be congruent to 0 (mod r_j) so that the projection along $i_{d+1} = 0$ is in the same position as that for $i_{d+1} = \beta\delta$. Summing the j^{th} coordinate over all d-tuples $\vec{I} \in \Pi[r_j]$ we get $\frac{r}{r_j} \sum_{l=0}^{r_j - 1} l = \frac{r}{r_j}\binom{r_j}{2}$. In U the term corresponding to each d-tuple appears $r^{\eta-2} = \delta/r$ times. So the j^{th} coordinate of $G(\beta\delta - 1)$ is $\frac{\delta}{2}(r_j - 1)$. This is congruent to 0 modulo r_j unless r_j is even, the rest of the r_l are odd and $\eta = 2$.

Let $V(s) = ([V]_s, [V]_{s+1}, \ldots, [V]_{s+\eta-1})$ and let $U(s) = ([U]_s, [U]_{s+1}, \ldots, [U]_{s+\eta-2})$. That is, $V(s)$ is the length η string in V starting at position s and $U(s)$ is the length $\eta - 1$ string in V starting at position s. Because $\gcd(\beta, \delta) = 1$, the pairs $(V(s), U(s))$ are distinct for $s = 0, 1, \ldots, \beta\delta - 1$. That is, each length η string from V and length $\eta - 1$ string from U appear together uniquely in position s for some $0 \leq s < \beta\delta$.

Finally, to see that B is De Bruijn, consider any k-ary order $\vec{N}'$ matrix M. Let $M[h]$ be the order $\vec{N}$ submatrix projecting along $i_{d+1} = h$. $M[h]$ appears uniquely in $\mathcal{F}$. Say it appears in position $\vec{J}(h)$ in $F_{z(h)}$. The sequence $(\vec{J}(1) - \vec{J}(0), \vec{J}(2) - \vec{J}(1), \ldots, \vec{J}(\eta - 1) - \vec{J}(\eta - 2))$ corresponds to a length $\eta - 1$ sequence appearing uniquely in U via the bijection g. The sequence $(z(0), z(1), \ldots, z(\eta - 1))$ appears uniquely in V. These two sequences appear together in position s for exactly one $0 \leq s < \beta\delta$ since $\gcd(\beta, \delta) = 1$. Then M appears in B in position $\vec{I}$, where $i_{d+1} = s$ and the first d coordinates of $\vec{I}$ are given by $G(s - 1) + \vec{J}(0)$.

The constructions for the other cases are similar but we have to work a bit harder to get the property that each pair $(V(s), U(s))$ appears uniquely in a string of pairs with period $\langle \beta\delta \rangle$. In the previous case, this string (although not explicitely mentioned) had first coordinate given by V and second coordinate given by U. We will use a technique similar to that used by Fan et al. [3] and generalized in Hurlbert and Isaak [5].

Consider the case that $\gcd(\beta, \delta - 1) = 1$. The case $\gcd(\beta - 1, \delta) = 1$ is similar. Assume, without loss of generality, that the string of $\eta - 1$ zeroes appears in position 0 in U. Let U' be the string of length $\beta - 1$ consisting of the fundamental block of U with the first zero removed. Let W be a string of period $\langle \beta\delta \rangle$ with fundamental block starting with β zeroes followed by β copies of U'. That is, $[W]_j = 0$ if $0 \leq j < \beta$. For $0 \leq m \leq \delta - 2$ and $0 \leq j < \beta - 1$ let $[W]_{\beta + m(\delta-1)+j} = [U]_{j+1}$. It is not difficult to check that the pairs $(V(s), W(s))$ are distinct for $s = 0, 1, \ldots, \beta\delta - 1$. Simply replace U with W in the construction and proof for the case $\gcd(\beta, \delta) = 1$. $\square$

Observe that if the base k is a prime, then the r_i are powers of k and hence t is a power of k. Then the condition $\gcd(\beta-1,\delta)=1$ of the previous theorem is satisfied. Also, when $t=1$, the condition $\gcd(\beta,\delta)=1$ is satisfied and we get the result that Cock [1] noted without proof.

4 Cycle Decompositions

In the previous section we described how to construct $d+1$-dimensional tori from d dimensional tori given a certain decomposition of a De Bruijn array. In this section we describe a general procedure for obtaining such decompositions when $d=1$. The period and the size of the family $\mathcal{F}$ will depend on an initial choice of a De Bruijn cycle. We will look at one particular case to construct a family of 2-dimensional tori.

Definition 4.1. *Let $A=(a_0,a_1,a_2,\ldots,)$ be a k-ary infinite periodic sequence. The* difference integral *of A with respect to $c\in\{0,1,\ldots,k-1\}$, denoted $\Gamma(A;c)$, is the string $(c,c+a_0,c+a_0+a_1,\ldots,c+\sum_{j=0}^{i-1}a_j,\ldots)$ with addition performed modulo k. That is, the initial term of $\Gamma(A;c)$ is $[\Gamma(A;c)]_0=c$ and the j^{th} term $[\Gamma(A;c)]_j$ is $[\Gamma(A;c)]_{j-1}+a_{j-1}$.*

For example, (1) shows the $6-$ary fundamental block $A=(0,1,2,3,4,5)$ repeated twice and several of its difference integrals.

$$\begin{array}{c||cccccccccccc} A= & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline \Gamma(A;0)= & 0 & 0 & 1 & 3 & 0 & 4 & 3 & 3 & 4 & 0 & 3 & 1 \\ \Gamma(A;1)= & 1 & 1 & 2 & 4 & 1 & 5 & 4 & 4 & 5 & 1 & 4 & 2 \\ \Gamma(A;2)= & 2 & 2 & 3 & 5 & 2 & 0 & 5 & 5 & 0 & 2 & 5 & 3 \end{array} \tag{1}$$

It is easy to check that $\Gamma(A;3)$ is a cyclic shift of $\Gamma(A;0)$, $\Gamma(A;4)$ is a cyclic shift of $\Gamma(A;1)$, and $\Gamma(A;5)$ is a cyclic shift of $\Gamma(A;2)$. The relation $\Gamma(A;c+3)=\Gamma(A;c)^6$ holds because $\gcd(6,3)=3$.

Lemma 4.2. *Let $\mathcal{A}=\{A_0,A_1,\ldots,A_{t-1}\}$ be a k-ary, 1-dimensional De Bruijn family of order $\langle n\rangle$ and period $\langle r\rangle$. If there is an $x\in[k]$ such that $\sum_{l=0}^{r-1}[A_j]_l\equiv x \pmod k$ for all $j\in[t]$, then, for $\gamma=\gcd(k,x)$, $\Gamma(\mathcal{A})=\{\Gamma(A_j;c)|j\in[t];c\in[\gamma]\}$ is a k-ary, 1-dimensional De Bruijn family of order $\langle n+1\rangle$ and period $\langle rk/\gamma\rangle$.*

Proof. From the definition of difference integral we use

$$[\Gamma(A_j;c)]_s\equiv[\Gamma(A_j;c)]_{s-1}+[A_i]_{s-1} \pmod k \text{ for } s=1,2,\ldots \tag{2}$$

with $[\Gamma(A_j;c)]_0=c$. Thus the period of the strings in $\Gamma(\mathcal{A})$ is a multiple of the period $\langle r\rangle$ of strings in $\mathcal{A}$. From (2), observe also that $\Gamma(A_j;c)\neq\Gamma(A_j';c')$ for $j\neq j'$ and any c,c'.

Again from the definition of difference integral,

$$[\Gamma(A_j;c)]_s + \sum_{l=s}^{s'-1} [A_j]_l \equiv [\Gamma(A_j;c)]_{s'} \pmod k \text{ for } 0 < s < s'. \tag{3}$$

To find the period of $\Gamma(A_j;c)$ we seek the smallest m such that

$$[\Gamma(A_j;c)]_s \equiv [\Gamma(A_j;c)]_{s+mr}$$

for all s (since the period is a multiple of $\langle r \rangle$). By (3) and the choice of x this occurs if $mx \equiv 0 \pmod k$. The smallest such m is k/γ and so the period is $\langle rk/\gamma \rangle$.

We also have

$$[\Gamma(A_j;c)]_s + (c'-c) \equiv [\Gamma(A_j;c')]_s \pmod k \text{ for } s = 0, 1, \ldots. \tag{4}$$

Hence, by (3) and (4), we have $\Gamma(A;c+x) = \Gamma(A;c)^r$ and $\Gamma(A;c+\gamma) = \Gamma(A;c)^w$, where $wx \equiv \gamma \pmod{\frac{k}{\gamma}}$ (w is unique since $\gcd(\frac{x}{\gamma}, \frac{k}{\gamma}) = 1$).

Finally, we must show that if $v = (v_0, v_1, \ldots, v_n)$ is a $k-$ary string then it appears uniquely in exactly one of the strings in $\Gamma(\mathcal{A})$. Let $\Delta v = (v_1 - v_0, v_2 - v_1, \ldots, v_n - v_{n-1})$. Then v appears in position s in $\Gamma(A_j;c)$ only if (by (2)), Δv appears in position s of A_j and $v_0 = c + \sum_{l=0}^{s-1}[A_j]_l$. Since $\mathcal{A}$ is a De Bruijn family, there is only one such j. Let $c = v_0 - \sum_{l=0}^{s-1}[A_j]_l$ and let $c' < \gamma$ satisfy $c \equiv c' \pmod \gamma$. Then since $\Gamma(A_j;c)$ is a shift of $\Gamma(A_j;c')$, v appearing uniquely in $\Gamma(A_j;c)$ implies v appears uniquely in $\Gamma(A_j;c')$. □

Now for all $i > 1$ let

$$\Gamma(A;c_1,c_2,\ldots,c_i) = \Gamma\left(\Gamma(A;c_1,c_2,\ldots,c_{i-1});c_i\right)$$

and for a family $\mathcal{A}$ of strings, let

$$\Gamma^{(i)}(\mathcal{A}) = \left\{\Gamma(A;c_1,c_2,\ldots,c_i) | A \in \mathcal{A}, c_l \in [k], l = 1,2,\ldots,i\right\}.$$

Observe in (1) that each entry of $\Gamma(A;0)$ is the sum (mod 6) of the entry to the left and the entry above and to the left. This Pascal property, as stated in (2), allows the computation of arbitrary terms as in the following lemma.

Lemma 4.3.

$$[\Gamma(A;c_1,c_2,\ldots,c_i)]_s \equiv \sum_{l=0}^{s-1} \binom{s-1-l}{i-1} x_l + \sum_{j=1}^{i} \binom{s}{i-j} c_j \pmod k.$$

Proof. By induction. The base cases $i = 1$ or $s = 0$ are straightforward to check. Then,

$$\begin{aligned}
[\Gamma(A; c_1, c_2, \ldots, c_i)]_s &\equiv [\Gamma(A; c_1, c_2, \ldots, c_i)]_{s-1} + [\Gamma(A; c_1, c_2, \ldots, c_{i-1})]_{s-1} \\
&\equiv \sum_{l=0}^{s-2} \binom{s-2-l}{i-1} x_l + \sum_{j=1}^{i} \binom{s-1}{i-j} c_j \\
&\quad + \sum_{l=0}^{s-2} \binom{s-2-l}{i-2} x_l + \sum_{j=1}^{i-1} \binom{s-1}{i-j-1} c_j \\
&= \sum_{l=0}^{s-2} \binom{s-1-l}{i-1} x_l + c_i + \sum_{j=1}^{i-1} \binom{s}{i-j} c_j \\
&= \sum_{l=0}^{s-2} \binom{s-1-l}{i-1} x_l + + \sum_{j=1}^{i} \binom{s}{i-j} c_j \pmod{k}.
\end{aligned}$$

□

We now consider the strings obtained by iterating Γ starting with $0, 1, \ldots$ as in (1). Iterating Γ with different initial string will produce different results in terms of periods.

Theorem 4.4. *Let k have prime factorization $k = \Pi p_l^{\alpha_l}$ and let $A \in dB_k^1(k, 1)$ have fundamental block $(0, 1, 2, \ldots, k-1)$. Then $\Gamma^{(i)}(\mathcal{A})$ is a De Bruijn family of order $\langle i+1 \rangle$ and period $\langle k \Pi p_l^{\lfloor \log_{p_l}(i+1) \rfloor} \rangle$, where $\mathcal{A} = \{A\}$.*

Proof. It is enough to prove the theorem for $k = p^\alpha$ a prime power. This follows by an argument similar to the term product construction of Theorem 2.2. We omit the details as they are nearly identical to Theorem 2.2.

Assume $k = p^\alpha$ for p a prime. The proof is by induction on i. Now $[A]_0 + [A]_1 + \cdots + [A]_{k-1} = 0 + 1 + \ldots (k-1) = k(k-1)/2$, and so by Lemma 4.2, $\Gamma(\mathcal{A})$ is a De Bruijn family of order $\langle 2 \rangle$. The period depends on the parity of k. If k is odd, $k(k-1)/2 \equiv k \pmod{k}$ and the period is $\langle k \rangle$. If k is even, $k(k-1)/2 \equiv k/2 \pmod{k}$ and the period is $\langle 2k \rangle$.

Now assume by induction that the result holds for $\Gamma^{(i-1)}(\mathcal{A})$. Let $u = \lfloor \log_p i \rfloor$. Then the period of $\Gamma^{(i-1)}(\mathcal{A})$ is $\langle r \rangle = \langle p^{\alpha+u} \rangle$. Observe that

$$\sum_{h=0}^{r-1} [\Gamma(A; c_1, c_2, \ldots, c_{i-1})]_h \equiv [\Gamma(A; c_1, c_2, \ldots, c_{i-1}, 0)]_r \pmod{k} \tag{5}$$

by (3). By Lemma 4.2 and (5), we need to check that there exists an x such that $[\Gamma(A; c_1, c_2, \ldots, c_{i-1}, 0)]_r \equiv x \pmod{k}$ for all choices of $c_1 c_2 \ldots c_{i-1}$ and we must determine x.

Now, by the choice of A and by Lemma 4.3

$$\begin{aligned}[\Gamma(A; c_1, c_2, \ldots, c_{i-1}, 0)]_r &\equiv \sum_{l=0}^{r-1} \binom{r-1-l}{i-1} l + \sum_{j=1}^{i} \binom{r}{i-j} c_j \qquad (6)\\ &= \binom{r}{i+1} + \sum_{j=1}^{i-1} \binom{r}{i-j} c_j.\end{aligned}$$

Here we have used a basic combinatorial identity and $c_i = 0$.

For $(i-j) = 1, 2, \ldots, i-1 \leq p^{u+1} - 2$, p^α divides $\binom{r}{i-j}$ since $r = p^{\alpha+u}$. Hence each term in in the sum on the right side of (6) is congruent to 0 modulo $k = p^\alpha$. So x exists. (The sum of the first r terms of $\Gamma(A; c_1, c_2, \ldots, c_{i-1})$ is independent of the choice of the c_l. Indeed, x exists and is independent of the choice of the c_l for any initial choice of A.) We have

$$x \equiv \binom{r}{i+1} \equiv \binom{p^{\alpha+u}}{i+1} \pmod{k}.$$

If $p^u \leq i \leq p^{u+1} - 2$ this is equivalent to 0 modulo p^α and the period for $\Gamma^{(i)}(\mathcal{A})$ remains $\langle p^{\alpha+\lfloor \log_p(i-1+1) \rfloor} \rangle = \langle p^{\alpha+\lfloor \log_p i \rfloor} \rangle = \langle p^{\alpha+\lfloor \log_p(i+1) \rfloor} \rangle$. If $i = p^{u+1} - 1$ then $\binom{p^{\alpha+u}}{i+1} = \binom{p^{\alpha+u}}{p^{u+1}}$, which is equivalent to $p^{\alpha-1}$ modulo p^α. Hence, by Lemma 4.2, the period is $\langle p^{\alpha+\lfloor \log_p(i-1+1) \rfloor} \frac{p^\alpha}{p^{\alpha-1}} \rangle = \langle p^{\alpha+1+\lfloor \log_p i \rfloor} \rangle = \langle p^{\alpha+\lfloor \log_p(i+1) \rfloor} \rangle$. □

Now we can apply Theorem 3.3 to the De Bruijn family obtained in the previous Theorem to get new De Bruijn tori. Observe that the sizes are rarely powers of k. For example, using the strings of (1) we obtain $(12, 3 \cdot 36^{n-1}; 2, n)^2_6$ De Bruijn tori for every $n \geq 2$.

Theorem 1.1. *For all natural numbers n_1, n_2, k there exists a $(q, k^{n_1 n_2}/q; n_1, n_2)^2_k$ De Bruijn tori, where k has prime decomposition $k = \Pi p_l^{\alpha_l}$ and $q = k \Pi p_l^{\lfloor \log_{p_l} n_1 \rfloor}$.*

Proof. By Theorems 3.3 and 4.4. □

Observe that an alternative proof uses Theorem 4.4 only for prime powers along with the term product construction of Theorem 2.2.

Although this does yield De Bruijn tori whose periods are not powers of k, this still leaves open the challenge of producing tori whose periods are not multiples of k, such as $(16, 81; 2, 2)^2_6$.

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