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The reversing number of a digraph

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Abstract

A minimum reversing set of a digraph is a smallest sized set of arcs which when reversed makes the digraph acyclic. We investigate a related issue: Given an acyclic digraph D, what is the size of a smallest tournament T which has the arc set of D as a minimum reversing set? We show that such a T always exists and define the reversing number of an acyclic digraph to be the number of vertices in T minus the number of vertices in D. We also derive bounds and exact values of the reversing number for certain classes of acyclic digraphs.

1. Introduction

Recall that a tournament is a directed graph such that for each pair x, y of vertices exactly one of the arcs (x, y) or (y, x) is present. Slater [32] and Younger [38] introduced the study of minimum sized sets of arcs which when reversed make a tournament acyclic. Call such a set a *minimum reversing set*. As we shall observe, minimum reversing sets are related to other kinds of sets studied in the literature of electrical engineering, statistics, and mathematics. These are feedback arc sets, minimum sets of inconsistencies in a preference ordering, cycle transversals, and sets of consistent arcs in a tournament. We investigate a related issue: Given an acyclic digraph D, what is the size of a smallest tournament T which has the arc set of D as a minimum reversing set? The *reversing number* of D is the number of "extra vertices" in T.

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We shall adopt the graph theoretic notation that is summarized at the end of this section. If $F = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ is a set of arcs in a digraph, then its *reversal* is $F^R = \{(y_1, x_1), \ldots, (y_k, x_k)\}$. All digraphs considered in this paper will be simple; there are no parallel arcs between two vertices.

With this notation, we make the following definitions.

Definition 1. *A reversing set* of a tournament T is a set of arcs F, such that $(T \setminus F) \cup F^R$ is acyclic. A *minimum reversing set* in T is a reversing set of minimum size.

The notation $(T \setminus F) \cup F^R$ will be used often and indicates the tournament obtained by reversing the direction of the arcs in the set F.

Define an *ordering* on a tournament T on n vertices as a function σ from the vertex set of T to the set $\{1, 2, \ldots, n\}$. An ordering σ is said to be *acyclic* when $\sigma(x) < \sigma(y)$ whenever (x, y) is an arc of T. Since an acyclic tournament has a unique acyclic ordering (see e.g. [26]) that is in fact a linear order, we will talk about *the* acyclic order or *the* order obtained after reversing the arcs of a reversing set. Given a general ordering σ of the vertices of a tournament, we define the set of *backwards arcs* relative to σ to be the set of arcs (v, w) in the tournament such that $\sigma(w) < \sigma(v)$. With this notation, a reversing set F is the set of backwards arcs relative to the acyclic ordering of the tournament obtained by reversing the arcs in F.

Definition 2. Given an acyclic digraph D, the *reversing number* $r(D)$ of D is $\min(|V(T)| - |V(D)|)$, where the minimum is taken over all tournaments T such that D is a minimum reversing set of T.

We show in Theorem 7 that the reversing number is well defined if and only if D is an acyclic digraph, justifying the definition.

In our study of reversing numbers we will make use of results on minimum reversing sets. Reversing sets have been studied by a number of authors in different contexts using different terminologies. In the electrical engineering literature *feedback arc sets*, sets of arcs whose removal makes a digraph acyclic, have been studied. Given a digraph D, it is easy to see that a minimum set of arcs whose removal makes D acyclic is also a minimum set of arcs whose reversal makes D acyclic and vice versa, so the minimum feedback arc set problem and the minimum reversing set problem are equivalent. To see this, note that it is obvious that any set of arcs whose reversal creates an acyclic digraph also creates an acyclic digraph by its removal (since the remaining arcs form an acyclic digraph). Conversely, let F be a minimal subset of the arc set A of a tournament whose removal makes the tournament acyclic. By minimality, if $(x, y) \in F$, then (x, y) is contained in a cycle $C = (y, v_1, \ldots, v_k, x)$ in $(A \setminus F) \cup \{(x, y)\}$. If there is a cycle C' in the tournament $(A \setminus F) \cup F^R$ obtained by reversing the arcs of F, then replace each arc $(y, x) \in F^R$ which is on C' with the path $y, v_1, \ldots, v_k, x$ from a cycle C containing (x, y) in $(A \setminus F) \cup \{(x, y)\}$. Since all these arcs are in $A \setminus F$, this results in a closed directed chain in $A \setminus F$. Such a chain contains a cycle, contradicting the fact that removal of F creates an acyclic digraph. Thus, the equivalence is established.

Runyon first suggested study of the feedback arc set problem. (His question is cited in the list of problems in [31] and is called the feedback cut set problem.) Tucker [36] gave an integer programming formulation and Younger [38] began the analysis of the structure of the feedback arc sets. Lawler [23] formulated the problem of finding a minimum feedback arc set as a quadratic assignment problem. Hakimi [13], Lempel and Cederbaum [24], Kamae [19], and Yau [37] continued analysis of the structure of these sets and suggested algorithms and heuristics for finding minimum feedback arc sets in general. In addition, Karp [20] showed that finding the size of a minimum reversing set, i.e. a minimum feedback arc set, is NP-hard in general.

In the statistics literature, Slater [32] first suggested the study of *minimum sets of inconsistencies* of a preference ordering (ranking) with the observed relations from a complete paired comparison experiment. The graph theoretic model of paired comparison experiments has the objects being compared as vertices of a digraph and an arc from x to y if and only if x is preferred to y. A *nearest adjoining order* is a linear order such that the number of preferences inconsistent with that order is minimized. Since preferences in a linear order induce an acyclic tournament, minimizing the set of inconsistencies is the same as finding a minimum set of arcs whose reversal makes the preference digraph acyclic and vice versa. Slater [32] sought to determine the probability distribution over every tournament (outcomes of all possible comparisons) of the size of a minimum set of inconsistencies over all possible orderings. This work was continued by Alway [1], Thompson and Remage [35], Remage and Thompson [29], Bermond [4], Bermond and Kodratoff [6], Monjardet [25], Hubert [16], and Baker and Hubert [2], to name a few, with suggestions for algorithms and study of more general questions with different weightings on the amount of inconsistency. Ref. [16] is a survey uniting the electrical engineering and statistics literature.

A third source of interest in minimum reversing sets arises in the mathematics literature. Erdős and Moon [10] introduced the question of finding the greatest integer k such that every tournament on n vertices has a set of k consistent arcs (i.e., an acyclic subdigraph with k arcs). The study of this value has been continued by Reid [27], Reid and Parker [28], Spencer [33, 34], and de la Vega [9]. A number of authors have studied the computational aspects of determining a largest acyclic subdigraph of a digraph. The complement in a digraph of the arc set of a largest acyclic subdigraph is a minimum reversing set of the digraph and vice versa. The polytope of the largest acyclic subdigraph problem has been studied by Grötschel et al. [11, 12] and Jünger [18]. Korte [21] examines approximation algorithms for this problem.

As we have already remarked, the problems mentioned above are all equivalent. (This has been proved by a number of authors.) Since reversing the arcs in a minimum reversing set makes a digraph acyclic, every cycle in the digraph must contain an arc from the minimum reversing set. That is, the arcs of a minimum reversing set are a *transversal* of the cycles in the digraph. In fact the minimum size of a transversal of cycles in a digraph is equal to the size of a minimum reversing set. (This follows from the fact that removing the arcs of a transversal of cycles creates an acyclic digraph and from the equivalence of minimum feedback arc sets and minimum reversing sets.) This has been shown by Dambit and Gindberg (cited in [5]) and Remage and Thompson [29]. All of this can be summarized by the following.

Remark. In a tournament, the problems of finding a minimum reversing set, a minimum set of inconsistencies, a minimum feedback arc set, a largest acyclic subdigraph, and a minimum transversal of cycles are all equivalent.

See [18] for more information on equivalent versions of the problem of finding a minimum reversing set and for applications.

Since a minimum reversing set is also a minimum transversal of the cycles, every arc in a minimum reversing set is contained in a cycle. In fact, we show in Theorem 6 that every arc of a minimum reversing set in a tournament must be contained in some cycle on three vertices (a 3-cycle). However, while the largest collection of arc disjoint cycles in a digraph provides a nice lower bound on the size of a minimum reversing set, this bound is not tight. Kotzig [22] and Bermond and Kodratoff [6] have shown that for $n \geqslant 10$ the bound is not tight even for tournaments, i.e., for $n \geqslant 10$ there exist tournaments on n vertices such that the size of a minimum reversing set is strictly greater than the largest collection of disjoint cycles in the tournament (see also [7]).

In Section 2, we review basic results on reversing sets which are useful in the study of reversing numbers. We also show that the reversing number is well defined. In Section 3, we develop some basic bounds on the reversing number. In particular, we show that the reversing number of an acyclic tournament on n vertices is an upper bound on the reversing number of any acyclic digraph on n vertices. A *Hamiltonian* path in a digraph is a directed path which meets every vertex in the digraph once. We also show a lower bound of $n-1$ on the reversing number of an acyclic digraph on n vertices if the digraph contains a Hamiltonian path. Graphs with reversing number 0 are studied in Section 4. Using a technique to extend a digraph on n vertices to a digraph on $n+1$ vertices without increasing the reversing number, we show that there are connected acyclic digraphs with reversing number 0 for $n \geqslant 7$. A parameter $d(n,r)$ giving the size of the largest arc set of an acyclic digraph on n vertices with reversing number r is also introduced in Section 4. Bounds on $d(n,1)$ and $d(n,0)$ are examined. Section 5 shows that the reversing number of an acyclic tournament on n vertices is between $2n - 4\log_2 n$ and $2n-4$. Finally, Section 6 establishes bounds on the reversing number of arborescences and exact values of the reversing number for directed stars, disjoint arcs, alternating paths, complete bipartite digraphs, alternating cycles.

We use the following graph theoretic notation. Any terms not defined here can be found in [14] or [26]. A *digraph* $D = (V(D), A(D))$ is a set of vertices $V(D)$ and arcs $A(D)$ which are ordered pairs from $V(D)$. For an arc (x,y), x will be called the *tail* and

y the *head*. The *outdegree* $d_D^+(v)$ of a vertex v in a digraph D is the number of arcs in D in which v is the tail. For simplicity in notation we will use D to denote the arc set $A(D)$ when there is no chance of confusion. $D|_X$ will denote the subdigraph of D induced by the vertices of $X \subseteq V(D)$. Recall that the *reversal* of a set of arcs A is the set of arcs $A^R = \{(v,w) \mid (w,v) \in A\}$. A digraph will be called *connected* if the underlying graph is connected. A *cycle* in a digraph is a sequence of arcs $(v_0, v_1), (v_1, v_2), \ldots, (v_k, v_0)$ with all vertices distinct. Such a cycle will be denoted by $(v_0, v_1, \ldots, v_k)$ and called a $(k+1)$-*cycle*. An *acyclic* digraph contains no cycles; D is acyclic if and only if there is an ordering π such that $(x,y) \in D \Rightarrow \pi(x) < \pi(y)$. Such an ordering will be called an *acyclic ordering*. A *source* (*sink*) in a digraph is a vertex with no incoming (outgoing) arcs. A *tournament* T is a digraph such that for each pair $\{x,y\} \in V(T)$ exactly one of (x,y) or (y,x) is in T. A tournament is acyclic if and only if it has no 3-cycle. Throughout the text, we shall assume that the digraphs are without isolated vertices.

2. Basic results on minimum reversing sets

The following lemmas regarding reversing sets will be useful in the study of reversing numbers. The first three are from [38]. All follow easily from the definitions above.

Lemma 1 (Younger [38]). *If F is a minimum reversing set of a tournament T then, for $F' \subseteq F$, F' is a minimum reversing set of $T' = (T \setminus B) \cup B^R$ where $B = F \setminus F'$.*

Lemma 2 (Younger [38]). *If a vertex v is a source or sink in a tournament T, then F is a minimum reversing set of T if and only if F is a minimum reversing set of $T \setminus \{v\}$.*

Recall that an acyclic tournament has a unique acyclic ordering.

Lemma 3. (Younger [38]). *If T is a tournament and F is a minimum reversing set such that $\pi(v_1) < \pi(v_2) < \cdots < \pi(v_n)$ is the acyclic ordering after reversal of the arcs in F, then for any segment $v_i, v_{i+1}, \ldots, v_{i+j} = S$, $F|_S$ is a minimum reversing set of $T|_S$.*

Lemma 1 says that if F is a minimum reversing set of a tournament T then for any subset F' of F, if we reverse in T the arcs which are in F but not in F' the new tournament T' has F' as a minimum reversing set. If T' had a smaller reversing set B then $(F \setminus F') \cup B$ would be a reversing set of T smaller than F. Lemma 2 states that no arc in a minimum reversing set of a tournament T has a tail which is a source in T or a head which is a sink in T. Lemma 3 is a direct consequence of Lemmas 1 and 2.

Lemma 4. *If T is a tournament and W is any subset of the vertices of T, then for a minimum reversing set F of T, the number of arcs in F joining vertices of W is greater than or equal to the size of a minimum reversing set of $T|_W$.*

Proof. The arcs in F with both ends in W form a reversing set of T restricted to W. $\square$

Lemma 5. *If τ is a collection of arc disjoint cycles in a tournament T, then for each reversing set F in T,*

$$|\tau| \leqslant |F|.$$

Proof. If $C \cap F = \emptyset$ for a cycle C in T, then C is a cycle in $(T \backslash F) \cup F^R$, contradicting the assumption that F is a reversing set. So each cycle contains at least one arc from F. Since the cycles are arc disjoint the bound follows. $\square$

We have mentioned in the introduction that each arc of a minimum reversing set of a tournament T is in a 3-cycle of T. The proof of this is given in Theorem 6.

Theorem 6. *Let T be a tournament and let F be a minimum reversing set of T. Then every arc of F belongs to some 3-cycle of T.*

Proof. Consider an arc $(y, z) \in F$. Reversing the arcs of F which do not meet y or z will not affect inclusion of (y, z) in a 3-cycle of T. By Lemma 1, reversing these arcs does not affect inclusion of (y, z) in a minimum reversing set. Thus, it is enough to show the result for $(y, z) \in F$, F and T such that every arc of F is incident on either y or z. Assume that this is the case. Assume also that the vertices are labeled so that the acyclic ordering π of $(T \backslash F) \cup F^R$ is $\pi(x_1) < \pi(x_2) < \cdots < \pi(x_n)$. So every arc of F goes from x_j to x_k for $j > k$. Note that deleting vertices v such that $\pi(v) < \pi(z)$ or $\pi(v) > \pi(y)$ will not form new 3-cycles. Thus, we may assume that $(y, z) = (x_n, x_1)$. It also follows that every arc of F has the form (x_j, x_1) or (x_n, x_j) since arcs (x_j, x_i) for $1 < i < j < n$ do not meet $y = x_n$ or $z = x_1$.

For $k = 1, n$, let

$$X_k^+ = \{(x_k, x_j) \in T\colon 1 < j < n\},$$

$$X_k^- = \{(x_j, x_k) \in T\colon 1 < j < n\}.$$

Note that the four sets described above are all disjoint and that $F = X_1^- \cup X_n^+ \cup \{(x_n, x_1)\}$. Also, since all arcs of T which join x_i to x_j, $1 < i < j < n$, go from x_i to x_j, it follows that $[T \backslash (X_1^+ \cup X_n^-)] \cup (X_1^+ \cup X_n^-)^R$ is acyclic with acyclic ordering π' satisfying $\pi'(x_n) < \pi'(x_2) < \cdots < \pi'(x_{n-1}) < \pi'(x_1)$. Since F is a minimum reversing set, we have

$$|X_1^+| + |X_n^-| = |X_1^+ \cup X_n^-| \geqslant |F| = |X_1^-| + |X_n^+| + 1.$$

Thus, since $|X_1^+| + |X_1^-| + |X_n^+| + |X_n^-| = 2(n-2)$, we have $|X_1^+| + |X_n^-| > (n-2)$. By the pigeonhole principle, there exists a j with $1 < j < n$ such that both (x_1, x_j) and (x_j, x_n) are in $X_1^+ \cup X_n^- \subset T$. Then (x_1, x_j, x_n) is a 3-cycle in T containing (x_n, x_1). $\square$

We have noted that minimum reversing sets are necessarily acyclic. The next theorem shows that every acyclic digraph arises as a minimum reversing set of some tournament.

Theorem 7. *Let D be a digraph. The following two conditions are equivalent:*
(i) *D is acyclic.*
(ii) *D is a minimum reversing set of some tournament.*

Proof. If D contains a cycle then so does D^R; thus every reversing set must be acyclic. Conversely assume that D is acyclic. Assume also that the vertices $V(D) = \{u_1, u_2, \ldots, u_n\}$ are labeled so that there is an acyclic ordering π of D satisfying $\pi(u_1) < \pi(u_2) < \cdots < \pi(u_n)$. We construct a tournament T with minimum reversing set D as follows. Let $V(T) = V(D) \cup \{v_{ij}: (u_i, u_j) \in D\}$. Let T' be an acyclic tournament on $V(T)$ with acyclic ordering π' satisfying $\pi'(u_n) < \pi'(u_{n-1}) < \cdots < \pi'(u_1)$ and $\pi'(u_j) < \pi'(v_{ij}) < \pi'(u_i)$ for all v_{ij}. This can be done since $v_{ij} \in V(T) \Leftrightarrow (u_i, u_j) \in D \Rightarrow i < j$. Thus, corresponding to each arc (u_i, u_j) of D there is an extra vertex v_{ij} which falls between the ends of the arc in the ordering π'.

Note that $D^R \subseteq T'$, so we can define $T = (T' \setminus D^R) \cup D$, i.e., $T' = (T \setminus D) \cup D^R$. Since T' is acyclic, D is a reversing set of T. Also, $\tau = \{(u_i, u_j, v_{ij}): (u_i, u_j) \in D\}$ is a collection of arc disjoint 3-cycles in T with $|\tau| = |D|$. Therefore, by Lemma 5, D is a minimum reversing set of T. $\square$

It follows from Theorem 7 that the reversing number $r(D)$ is well defined and that

$$r(D) \leqslant |D|. \tag{1}$$

We use the notation $|D|$ to indicate the size of the arc set of D when there is no chance of confusion. This notation is consistent with the idea that we are viewing the arc sets of the digraphs as reversing sets.

Given an acyclic digraph D and tournament T, if D is a minimum reversing set of T and no tournament with fewer vertices than T has D as a minimum reversing set then we say that T *realizes* D. If T realizes an acyclic digraph D, then $r(D)$ is the number of vertices in $V(T) \setminus V(D)$. Observe also that if D is an acyclic digraph, T a tournament that realizes D and σ an acyclic ordering of $(T \setminus D) \cup D^R$, then for every arc (x, y) of D, $\sigma(x) > \sigma(y) + 1$.

3. Basic results on reversing number

In this section we make use of basic results on minimum reversing sets to establish some elementary facts about the reversing number. We first get a bound on the reversing number of an acyclic digraph in terms of the reversing number of a tournament by using a more general bound on the reversing number of subdigraphs.

Theorems 8. *Let $D' \subseteq D$ be acyclic digraphs on n vertices. Then $r(D') \leqslant r(D)$.*

Proof. By Lemma 1, if T is a tournament having D as a minimum reversing set then there is a tournament T' on the same number of vertices having D' as a minimum reversing set. □

Note here that it is important that both D and D' have the same number of vertices; otherwise Theorem 8 is not true. For example a single arc has reversing number 1 (Theorem 13), but many nontrivial acyclic digraphs have reversing number 0 (Theorem 17).

Corollary 9. *For an acyclic digraph D on n vertices, we have $r(D) \leqslant r(T_n)$, where T_n is the acyclic tournament on n vertices.*

Theorem 18 will give some bounds on the reversing number of acyclic tournaments. These together with Corollary 9 will give general bounds on the reversing number of any acyclic digraph.

We next take note of several basic results for getting bounds on the reversing number of an acyclic digraph D.

Lemma 10. *For an acyclic digraph D, $r(D) = r(D^R)$.*

Proof. For any tournament T, $(T\backslash D)\cup D^R$ is acyclic if and only if $(T^R\backslash D^R)\cup D$ is acyclic. Thus D is a minimum reversing set of T if and only if D^R is a minimum reversing set of T^R. □

Lemma 11. *Let D be an acyclic digraph and let T realize D. If $\pi(v_1) < \pi(v_2) < \cdots < \pi(v_n)$ is the acyclic ordering of $(T\backslash D)\cup D^R$, then for any segment $S = v_i, v_{i+1}, \ldots, v_{i+j}$, the number of non-D vertices in S is greater than or equal to the reversing number of $D|_S$.*

Proof. By Lemma 3, $D|_S$ is a minimum reversing set of $T|_S$. Thus $T|_S$ has at least as many non-D vertices as a tournament realizing $D|_S$. □

Let D be an acyclic digraph with vertex set V. For some $v \in V$, suppose $V\backslash\{v\}$ can be partitioned as $V'_1 \cup V'_2$ such that in every acyclic ordering of D, the vertices of V'_1 come before v and the vertices of V'_2 come after v. Suppose also that there are no arcs from V'_1 to V'_2. Then v will be called an *order splitting vertex* of D and V'_1 is its *opening set* and V'_2 its *closing set*. By the definition of acyclic orderings, there are also no arcs from V'_2 to V'_1.

Lemma 12. *If v is an order splitting vertex of an acyclic digraph D, and V'_1 and V'_2 its opening and closing sets, respectively, then $r(D) = r(D_1) + r(D_2)$, where D_1 and D_2 are the digraphs induced by $V_1 = V'_1 \cup \{v\}$ and $V_2 = V'_2 \cup \{v\}$, respectively.*

Proof. Let T realize D and π be an acyclic ordering of $(T\backslash D)\cup D^{\mathrm{R}}$. Let W_1 be those vertices x of $V(T)$ with $\pi(x) \geqslant \pi(v)$ and W_2 the vertices x of $V(T)$ with $\pi(x) \leqslant \pi(v)$. Note that v is in both of these sets and that $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$. By Lemma 11, $r(D_1) \leqslant |W_1\backslash V_1|$ and $r(D_2) \leqslant |W_2\backslash V_2|$ and so $r(D_1) + r(D_2) \leqslant |W_1\backslash V_1| + |W_2\backslash V_2| = r(D)$.

To show the reverse inequality, we construct a tournament T' on $r(D_1) + r(D_2) + |V(D)|$ vertices having D as a minimum reversing set. Let T_1 realize D_1 and T_2 realize D_2. For $i = 1,2$ denote the vertex set of T_i by W_i. We can choose W_1 and W_2 so that $W_1\backslash\{v\}$ and $W_2\backslash\{v\}$ are disjoint. Then $(T_1\backslash D_1)\cup D_1^{\mathrm{R}}$ is an acyclic tournament. Let π' be the acyclic ordering of $(T_1\backslash D_1)\cup D_1^{\mathrm{R}}$ and let w denote the (unique) source in $(T_1\backslash D_1)\cup D_1^{\mathrm{R}}$. If $w \in V(T_1)\backslash V_1$, then by Lemma 3, D_1 is a minimum reversing set of $T_1\backslash\{w\}$, contradicting the assumption that T_1 realizes D_1. If $w \in V_1'$ then the reverse σ of the ordering on V_1 defined by π' is an acyclic ordering of D_1 for which v is not the last vertex. Since there are no arcs between V_1' and V_2' in D, we can combine σ with any acyclic ordering (with respect to $D_2|_{V_2'}$) of V_2' to follow σ. This gives an acyclic ordering of D for which not all the vertices of V_1' appear before v, contradicting the fact that V_1' is the opening set for the order splitting vertex v. Thus the source w in $(T_1\backslash D_1)\cup D_1^{\mathrm{R}}$ must be v. In a similar manner, it can be shown that $(T_2\backslash D_2)\cup D_2^{\mathrm{R}}$ is an acyclic tournament with v as a sink.

Let T' be the tournament formed by joining T_1 and T_2 at v with all arcs between T_1 and T_2 going from T_2 to T_1. Note that the arc set of T' can be partitioned into three parts, the arc set of T_1, the arc set of T_2, and the set of arcs between $W_1\backslash\{v\}$ and $W_2\backslash\{v\}$, all of which are directed from $W_2\backslash\{v\}$ to $W_1\backslash\{v\}$.

Since there are no arcs between V_1' and V_2', the arc set of D is partitioned into the arc set of D_1 and the arc set of D_2. So $D = D_1 \cup D_2$ and $|D| = |D_1| + |D_2|$ since these sets are disjoint. Consider $T = (T'\backslash D)\cup D^{\mathrm{R}}$. Since D_i is a reversing set of T_i for $i = 1,2$, $T|_{W_1}$ and $T|_{W_2}$ are acyclic. (This uses the fact that the arc sets of $T|_{W_1}$ and $T|_{W_2}$ are disjoint.) Since also all arcs in T between W_2 and W_1 are directed from W_2 to W_1, T is acyclic. Thus D is a reversing set of T'.

Finally, we show that every minimum reversing set of T' has size $|D|$, and thus that D is a minimum reversing set of T'. If F is a minimum reversing set of T', then $|F|_{W_1}| \geqslant |D_1|$ by Lemma 4 and the fact that D_1 is a minimum reversing set of $T'|_{W_1}$. Similarly, $|F|_{W_2}| \geqslant |D_2|$. Since the arc sets $F|_{W_1}$ and $F|_{W_2}$ are disjoint, $|F| \geqslant |F|_{W_1}| + |F|_{W_2}| \geqslant |D_1| + |D_2| = |D|$. The last equality follows since there are no arcs between V_1' and V_2' in D. Thus, D is a minimum reversing set of T' and $r(D) \leqslant |D_1| + |D_2|$. $\square$

Recall that the directed path P_n on n vertices is the digraph with vertex set $\{v_1, \ldots, v_n\}$ and arc set $\{(v_i, v_{i+1})\colon i = 1, \ldots, n-1\}$.

Theorem 13. *Let P_n be the directed path on n vertices. Then, $r(P_n) = n - 1$.*

Proof. A single arc P_2 has $r(P_2) = 1$ since it is not a minimum reversing set of itself (the only tournament on 2 vertices) and it is a minimum reversing set of a 3-cycle. By repeated application of Lemma 12 the result follows since every vertex of P_n except v_1 and v_n is order splitting. □

Corollary 14. *If D is an acyclic digraph on n vertices containing a directed Hamiltonian path, then $r(D) \geqslant n-1$.*

Proof. Apply Theorem 8 to the result of Theorem 13. □

Note that if a digraph has a unique acyclic ordering, then it contains a directed Hamiltonian path. Then by the corollary, a digraph on n vertices with a unique acyclic ordering has reversing number at least $n-1$. However, when there is not a unique acyclic ordering, the reversing number can be small. The next theorem states a necessary condition for the reversing number to be 0.

Theorem 15. *If $r(D)=0$, then D has at least two distinct sources and at least two distinct sinks.*

Proof. Let $V(D)=\{v_1,v_2,\dots,v_n\}$, let T realize D, and let π be the acyclic ordering of $T'=(T\setminus D)\cup D^R$. Note that $(v_i,v_j)\in D \Rightarrow \pi(v_i)>\pi(v_j)$. Since $r(D)=0$ and T' is acyclic, we may assume that $\pi(v_i)=i$, $i=1,2,\dots,n$. Thus, v_1 is a sink of D. If $(v_2,v_j)\in D$ then $j=1$. However, if $(v_2,v_1)\in D$ then by Lemma 3 applied to $v_1,v_2=S$, $D|_S=(v_2,v_1)$ is a minimum reversing set of the (acyclic) tournament on 2 vertices, a contradiction. Thus v_2 must also be a sink of D. By a similar argument there must be at least two distinct sources. □

4. Small reversing numbers

We will next consider the smallest reversing number among digraphs on n vertices. For $n\geqslant 2$, let $r_n=\min r(D)$, where the minimum is taken over all acyclic digraphs D on n vertices having no isolated vertices. Also for $n\geqslant 2$, let $r_n'=\min r(D)$, where the minimum is taken over all connected acyclic digraphs D on n vertices. Clearly we have $r_n\leqslant r_n'$, for every $n\geqslant 2$.

In order to calculate these parameters we introduce conditions under which extending certain digraphs will produce new digraphs without increasing the reversing number. These conditions also prove useful in examining the reversing number in general and for special classes of digraphs. Let D be an acyclic digraph, let T realize D and let τ be a collection of $|D|$ arc disjoint cycles in T. (Note that it is not necessary that a T realizing D contain such a collection τ.) Also let $S=\{(x_i,y_i)\in T$: $i=1,2,\dots,k\}$ be a collection of arcs from T none of which is an arc of one of the cycles in τ and assume that S is vertex disjoint, i.e., the x_i and y_i are all distinct. Let z be any element not in $V(T)$. We define two new digraphs: D', the *sink extension of D with* *respect to S*, and D'', the *source extension of D with respect to S*, as follows:

$$V(D') = V(D'') = V(D) \cup \{z\},$$

$$A(D') = A(D) \cup \{(z, x_i): i = 1, 2, \dots, k\},$$

$$A(D'') = A(D) \cup \{(y_i, z): i = 1, 2, \dots, k\}.$$

We also define T', the D' extension of T with respect to S, and T'', the D'' extension of T with respect to S, as follows. Let $M = \{x_1, \dots, x_k\} \cup \{y_1, \dots, y_k\}$ be the set of vertices which are endpoints of the arcs in S. Let T' and T'' have vertex sets $V(T') = V(T'') = V(T) \cup \{z\}$ and arc sets

$$A(T') = A(T) \cup \{(z, x_i), (y_i, z): i = 1, 2, \dots, k\} \cup \{(v, z): v \in V(T) \backslash M\},$$

$$A(T'') = A(T) \cup \{(z, x_i), (y_i, z): i = 1, 2, \dots, k\} \cup \{(z, v): v \in V(T) \backslash M\}.$$

Finally, we define the extensions τ' and τ'' of τ with respect to S by

$$\tau' = \tau'' = \tau \cup \{(x_i, y_i, z): i = 1, 2, \dots, k\}.$$

Lemma 16. *Let D be an acyclic digraph with reversing number $r(D)$, and let T realize D. Assume also that there is a collection τ of $|D|$ arc disjoint cycles in T and a set S of vertex disjoint arcs in T, none of which is an arc of a cycle from τ. Let D' be the sink extension of D with respect to S, T' be the D' extension of T with respect to S, and τ' the extension of τ with respect to S. Also, let D'' be the source extension of D with respect to S, T'' be the D'' extension of T with respect to S, and τ'' the extension of τ with respect to S. Then the following hold:*

(i′) *τ' is a collection of $|D| + |S|$ arc disjoint cycles in T',*
(ii′) *D' is a minimum reversing set of T',*
(iii′) *$r(D') \leqslant r(D)$,*
and
(i″) *τ'' is a collection of $|D| + |S|$ arc disjoint cycles in T'',*
(ii″) *D'' is a minimum reversing set of T'',*
(iii″) *$r(D'') \leqslant r(D)$.*

Proof. Let $S = \{(x_i, y_i) \in T: i = 1, \dots, k\}$. The cycles added to τ to obtain $\tau' = \tau''$ are arc disjoint from τ by the choice of S and since $z \notin V(T)$. Thus $|\tau'| = |\tau''| = |\tau| + |S|$. Also by the definitions of T', T'', τ', and τ'', each of the cycles in τ' is in T' and each of the cycles of τ'' is in T''. Thus (i′) and (i″) hold.

Note that $(T' \backslash D') \cup (D')^R$ is acyclic since $(T \backslash D) \cup D^R$ is acyclic, and that z is a sink in $(T' \backslash D') \cup (D')^R$. Analogously, $(T'' \backslash D'') \cup (D'')^R$ is acyclic with source z. Thus D' is a reversing set of T' and D'' is a reversing set of T''. By Lemma 5 applied to τ' and τ'', minimum reversing sets of T' and T'' have size at least $|\tau'|$ and $|\tau''|$, i.e., each has size at least $|\tau| + |S|$. Then, since D' is a reversing set of size $|D'| = |D| + |S| = |\tau| + |S| = |\tau'|$, D' is a minimum reversing set of T' and (ii') holds. Similarly, D'' is a minimum reversing set of T'' and (ii'') holds.

Note that $|V(T')| = |V(T'')| = |V(T)| + 1$. Since D' is a minimum reversing set of T', $r(D') \leqslant |V(T')| - |V(D')| = |V(T)| + 1 - (|V(D)| + 1) = r(D)$, and similarly $r(D'') \leqslant r(D)$. So (iii') and (iii'') hold. $\square$

This lemma also provides a foundation for dealing with various special classes of digraphs. While it is not difficult to construct digraphs with $n \geqslant 7$ vertices with reversing number 0, we will prove the result for alternating paths as an example of the use of Lemma 16 in dealing with special classes of digraphs considered in Section 6.

Determining r_n and r'_n for $n < 7$ requires some case analysis. In order to do this we review a result of Bermond and Kodratoff [6]. We look at the following upper bounds on the size of a minimum reversing set of a tournament on n vertices. Let m_n denote the maximum size of a minimum reversing set, where the maximum is taken over all tournaments on n vertices. Bermond and Kodratoff [6] show that $m_2 = 0$, $m_3 = m_4 = 1$, $m_5 = 3$, $m_6 = 4$, and $m_7 = 7$.

Theorem 17. *$r_2 = r'_2 = 1$; $r_3 = r'_3 = 2$; $r_4 = r'_4 = r_5 = r'_5 = 1$; $r_6 = 0$, $r'_6 = 1$ and for $n \geqslant 7$, $r_n = r'_n = 0$.*

Proof. We first consider cases when n is small.

Case $n = 2$: The only acyclic digraph on 2 vertices with no isolated vertices is an arc which is not a minimum reversing set of itself and is a minimum reversing set of a 3-cycle. Thus $r_2 = r'_2 = 1$.

Case $n = 3$: Every digraph on 3 vertices with no isolated vertices has at least two arcs and is connected. So $r_3 = r'_3$. Since $m_3 = m_4 = 1$ there is no tournament on 3 or 4 vertices having a connected digraph on three vertices as a minimum reversing set. Fig. 1 shows a tournament on five vertices, with a connected digraph on three vertices as a minimum reversing set, so $r_3 = r'_3 = 2$.

Case $n = 4$: An acyclic digraph on 4 vertices with no isolated vertex has at least 2 arcs. Since $m_4 = 1$, we have $r_4 \geqslant 1$ and $r'_4 \geqslant 1$. Fig. 2 shows a connected digraph on 4 vertices and a tournament realizing it, so $r_4 = r'_4 = 1$.

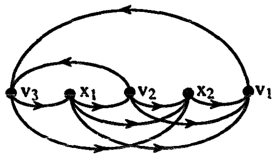


Fig. 1. A tournament realizing a connected digraph on three vertices, containing disjoint cycles (v_1, v_3, x_2) and (v_2, v_3, x_1).

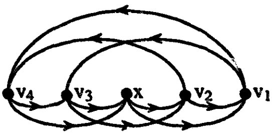


Fig. 2. A tournament on five vertices realizing a connected digraph on four vertices, containing disjoint cycles (v_2, v_4, v_3), (v_1, v_4, x) and (v_1, v_3, x, v_2).

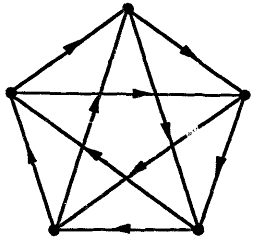


Fig. 3. A regular tournament on five vertices.

Case $n = 5$: Any acyclic digraph on 5 vertices with no isolated vertex has at least 3 arcs. Recall that the outdegree $d_T^+(x)$ of vertex x in T is the number of arcs $(x,j) \in T$. Consider any tournament T on 5 vertices. If some vertex x in T has outdegree 4 then x is a source and by Lemma 2, a minimum reversing set of T is a minimum reversing set of $T \backslash \{x\}$. Since $m_4 = 1$, the maximum size of a minimum reversing set of such a tournament is 1 and thus T cannot realize a digraph on 5 vertices containing no isolated vertex.

Consider tournaments T on 5 vertices having no vertex with outdegree 4 and some vertex x with $d_T^+(x) = 3$. Then reverse the arc for which x is the head to obtain a new tournament T' which has a vertex of degree 4 and, as above, a minimum reversing set of size at most 1. Thus T has a reversing set of size at most 2. Then a minimum reversing set of T has size at most 2 and T cannot realize a digraph on 5 vertices.

Finally, if T is a tournament on 5 vertices such that $d_T^+(x) \leqslant 2$ for all vertices x in T, then T is a regular tournament with all 5 vertices having degree 2. All such tournaments are isomorphic to the tournament shown in Fig. 3. It is straightforward to show that all of its minimum reversing sets have three connected arcs and hence contain an isolated vertex. Thus $r_5 \geqslant 1$ and $r'_5 \geqslant 1$. Fig. 4 gives an example to show that $r_5 = r'_5 = 1$.

Case $n = 6$: Fig. 5 shows that $r_6 = 0$. Any connected digraph on 6 vertices has at least 5 arcs and since $m_6 = 4$, no tournament on 6 vertices realizes a connected digraph on 6 vertices. Thus $r'_6 \geqslant 1$ and Fig. 6 gives an example to show that $r'_6 = 1$.

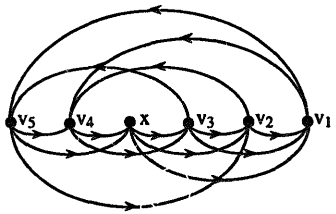


Fig. 4. A tournament on six vertices realizing a connected digraph on five vertices, containing arc disjoint cycles (v_2, v_4, v_3), (v_1, v_4, x), (v_1, v_5, v_2) and (v_3, v_5, x).

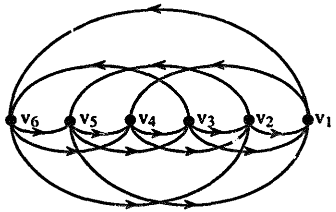


Fig. 5. A tournament on six vertices realizing a digraph on six vertices, containing arc disjoint cycles (v_3, v_6, v_5), (v_1, v_6, v_2), (v_2, v_5, v_4) and (v_1, v_4, v_3).

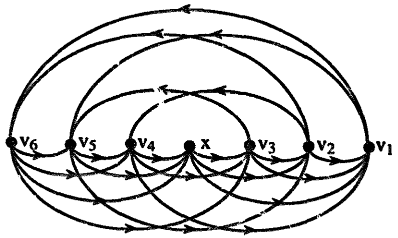


Fig. 6. A tournament on seven vertices realizing a connected digraph on six vertices, containing arc disjoint cycles (v_3, v_5, v_4), (v_2, v_4, x), (v_2, v_6, v_3), (v_1, v_5, x) and (v_1, v_6, v_4).

Case $n = 7$: We exhibit in Fig. 7 a connected acyclic digraph D_7 on 7 vertices, along with a T having D_7 as a reversing set, and a collection τ of $6 = |D_7|$ arc disjoint cycles in T. Thus $r(D_7) = 0$. This shows that $r_7 = r'_7 = 0$.

Case $n \geqslant 8$: We will show that alternating paths on n vertices, $n \geqslant 8$, have reversing number 0. An alternating path is a digraph based on a path graph. That is, an alternating path is the following digraph A_n or its reversal: $V(A_n) = \{v_1, \ldots, v_n\}$ and the arc set $A(A_n) = \{(v_i, v_{i-1}), (v_i, v_{i+1})$: i is odd, and both vertices are in $V\}$. Recall also that Lemma 10 says that $r(D) = r(D^R)$ for all D. Thus, in order to prove the result for all alternating paths it is enough to consider A_n.

By our convention of denoting the size of the arc set by $|A_n|$, we have $|A_n| = n - 1$. Fig. 8 exhibits a tournament $T(A_8)$ with A_8 as a reversing set. This tournament contains a set τ_8 of seven arc disjoint cycles and so, by Lemma 5, A_8 is a minimum reversing set of the tournament and $r(A_8) = 0$.

Note that $(v_8, v_4) \in T(A_8)$ and this arc is not an arc of any cycle in τ_8. Denoting the new vertex z in the sink extension by v_9, the sink extension of A_8 with respect to $S = (v_8, v_4)$ has vertex set $V(A_8) \cup \{v_9\}$ and arc set $A(A_8) \cup \{(v_9, v_8)\}$. Thus, the sink extension is A_9. By Lemma 16, $r(A_9) \leqslant r(A_8) = 0$. So $r(A_9) = 0$.

For $n \geqslant 9$ we prove by induction that there exist tournaments $T(A_n)$ and collections τ_n of arc disjoint cycles in $T(A_n)$ satisfying:

(a) $V(T(A_n)) = V(A_n)$.

(b) $T(A_n)$ has A_n as a minimum reversing set.

(c) $|\tau_n| = n - 1$.

(d) If n is odd, there is exactly one arc (v_n, v_{n-1}) in $T(A_n)$ with v_n as its tail, and for n even, there is exactly one arc (v_{n-1}, v_n) in $T(A_n)$ with v_n as its head.

(e) There is exactly one cycle in τ_n containing the vertex v_n. This is (v_{n-1}, x, v_n) if n is odd, and (x, v_{n-1}, v_n) if n is even for some $x \neq v_n, v_{n-1}$.

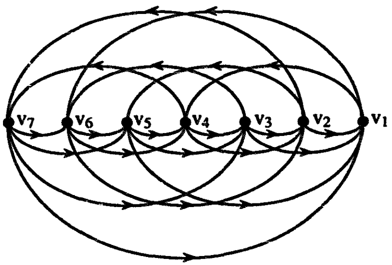


Fig. 7. A tournament on seven vertices realizing a connected digraph on seven vertices, containing arc disjoint cycles (v_1, v_6, v_5), (v_2, v_7, v_6), (v_4, v_7, v_5) (v_2, v_5, v_3), (v_3, v_6, v_4) and (v_1, v_4, v_2).

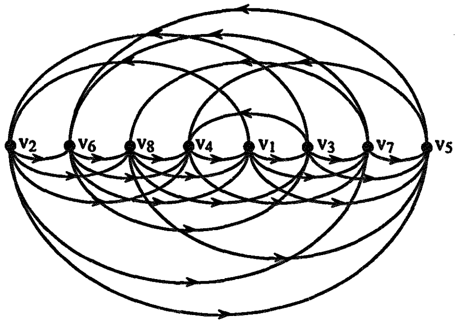


Fig. 8. A tournament realizing A_8 and the set $\tau_8 = (v_1, v_2, v_6), (v_3, v_2, v_8), (v_3, v_4, v_1), (v_5, v_4, v_7), (v_5, v_6, v_8)$ (v_7, v_6, v_3) and (v_7, v_8, v_1).

By (a), $|V(T(A_n))| = |V(A_n)|$. By (b), $r(A_n) \leqslant |V(T(A_n))| - |V(A_n)| = 0$. So, proving that (a) and (b) hold for all $n \geqslant 9$ will complete the proof.

Let $T(A_9)$ be the $D' = A_9$ extension of $T(A_8)$ and τ_9 the extension of τ_8, both with respect to (v_8, v_4). By the definition of the D' extension $T(A_9)$ and since $V(T(A_8)) = V(A_8)$, we have $V(T(A_9)) = V(A_9)$. So (a) holds. By Lemma 16, $T(A_9)$ has A_9 as a minimum reversing set. So (b) holds. Also, since $|\tau_8| = 7$ and by the definitions of the D' extension $T(A_9)$ and the (v_8, v_4) extension τ_9 of τ_8, it is easy to check that (c)–(e) hold for $n = 9$.

Assume by way of induction that the result holds for n. Consider $n + 1$ even (and thus n odd), $n + 1 \geqslant 10$. By (e), and since $|V(T(A_n))| \geqslant 3$, there exists a vertex $y \neq x$ which is not on the unique cycle $(v_{n-1}, x, v_n) \in \tau_n$ containing v_n. By (d), $(y, v_n) \in T(A_n)$ since $y \neq v_{n-1}$ and (v_n, v_{n-1}) is the only arc in the tournament with v_n as its tail. By (b), $r(A_n) \leqslant |V(T(A_n))| - |V(A_n)| = 0$. Since the reversing number is nonnegative, $r(A_n) = 0$ and thus $T(A_n)$ realizes A_n. By (c), $|\tau_n| = n - 1 = |A_n|$. Thus, $T(A_n)$ and τ_n satisfy the conditions necessary to take the source extension of A_n with respect to (y, v_n). This source extension D'' of A_n with respect to (y, v_n) is A_{n+1}. This follows since if we denote the new vertex in the extension by v_{n+1}, the new arc is (v_n, v_{n+1}) and since n is odd.

The $D'' = A_{n+1}$ extension $T(A_{n+1})$ of $T(A_n)$ has A_{n+1} as a minimum reversing set by Lemma 16. So (b) holds. By induction $V(T(A_n)) = V(A_n)$. Then by the definition of the $D'' = A_{n+1}$ extension, $V(T(A_{n+1})) = V(A_{n+1})$ and (a) holds.

Additionally, from the construction of the tournament $T(A_{n+1})$, this tournament contains exactly one arc (v_n, v_{n+1}) with v_{n+1} as its head. So (d) holds. Finally, the extension of τ_n with respect to (y, v_n) is $\tau_{n+1} = \tau_n \cup \{(y, v_n, v_{n+1})\}$ and the new cycle is arc disjoint from the cycles of τ_n. So, $|\tau_{n+1}| = |\tau_n| + 1 = n$. The last equality follows by induction. So (c) holds. By construction, τ_{n+1} has exactly one cycle (y, v_n, v_{n+1}) containing the new vertex v_{n+1}. Thus (e) holds.

In a similar manner, for $n+1$ odd, by (d) and (e) for n, there is a vertex y in $V(T(A_n))$ such that (v_n, y) is an arc in $T(A_n)$ and such that (v_n, y) is not contained in any cycle of τ_n. By (c) and the fact that (a) and (b) imply that $T(A_n)$ realizes A_n, the sink extension of A_n with respect to (v_n, y) is defined. Then this sink extension of A_n with respect to (v_n, y) is A_{n+1} and in a manner similar to the case when $n+1$ is even, it can be checked that the $D' = A_{n+1}$ extension $T(A_{n+1})$ of $T(A_n)$ and the extension τ_{n+1} of τ_n, both with respect to (v_n, y), satisfy (a)–(e). $\square$

An interesting question is to determine the largest number of arcs a connected digraph on n vertices with reversing number 0 can have. A similar question can be asked for reversing number r. To study this we introduce the parameter $d(n,r) = \max|A(D)|$, where the maximum is taken over all connected acyclic digraphs with $|V(D)| = n$ and $r(D) = r$. If no such D exists for a given n and r, then we say that $d(n,r)$ does not exist.

Since we are considering connected digraphs on n vertices, $n-1 \leqslant d(n,r) \leqslant \binom{n}{2}$. By Eq. (1), $d(n,r) \geqslant r$. Since a minimum reversing set of any tournament contains at most half the arcs in the tournament, $d(n,r) \leqslant \frac{1}{2}\binom{r+n}{2}$. Thus we get

$$\max\{r, n-1\} \leqslant d(n,r) \leqslant \min\left\{\frac{1}{2}\binom{r+n}{2}, \binom{n}{2}\right\}. \tag{2}$$

Corollary 9 and Theorem 18 (below) show that $d(n,r)$ is undefined for $r > 2n-4$. By Theorem 17, $d(n,0)$ is defined if and only if $n \geqslant 7$.

Let $f(n)$ be the largest k such that every tournament on n vertices contains an acyclic digraph with k arcs. It appears that upper bounds on $f(n)$ might provide graphs with reversing number 0 and a large number of arcs, since there exists some tournament with n vertices containing no acyclic digraph with $f(n)+1$ arcs, i.e., minimum reversing sets of this tournament have at least $\binom{n}{2} - f(n) - 1$ arcs. The upper bound $f(n) \leqslant \frac{1}{2}\binom{n}{2} + cn^{3/2}$, c constant, determined by Erdős and Moon [10] and Spencer [33], would then give digraphs with reversing number 0 and $\frac{1}{2}\binom{n}{2} - cn^{3/2}$ arcs. However, this reasoning does not necessarily work since we assume that our digraphs are connected and have no isolated vertices, while the digraphs obtained as minimum reversing sets of tournaments providing the upper bounds on $f(n)$ may have isolated vertices.

Making use of Lemma 16, we can show that for $n \geqslant 7$, $d(n,0) \geqslant \lceil (n-1)^2/\alpha \rceil$ where $\alpha = 5 + \sqrt{21}$ (see a preliminary version of this paper, [3]). As suggested by a referee, making use of Steiner triple systems, one can show $d(n,0) \geqslant n(n-1)/6$ at least for $n \equiv 1, 3 \pmod 6$. We are also able to show, using a particular "bipartite" digraph, that $(n^2+n)/4 \geqslant d(n,1) \geqslant (n^2+2n)/8$ (again, see [3]).

5. Acyclic tournaments

The reversing number of acyclic tournaments is important since it gives an upper bound on the reversing number of general digraphs as noted in Corollary 9.

Theorem 18. *For the acyclic tournament T_n on n vertices, $2n - 4\log_2 n \leqslant r(T_n) \leqslant 2n - 4$.*

Proof. In this proof, all logarithms will be base 2. Let T_n be an acyclic tournament with vertex set $V(T_n) = \{v_1, v_2, \ldots, v_n\}$ such that the acyclic ordering of T_n is $\pi'(v_n) < \pi'(v_{n-1}) < \cdots < \pi'(v_1)$.

In order to obtain a lower bound on the reversing number of the acyclic tournament T_n on n vertices we consider a smallest tournament $T(T_n)$ having T_n as a minimum reversing set. Since T_n is a minimum reversing set of $T(T_n)$, the acyclic order π of $T(T_n)$ after reversal of the arcs in T_n satisfies $\pi(v_1) < \pi(v_2) < \cdots < \pi(v_n)$. By Lemma 3 we may assume that for all vertices u in $T(T_n)$, $\pi(v_1) < \pi(u) < \pi(v_n)$ since otherwise there would be a smaller tournament having T_n as a minimum reversing set. Denote the extra vertices (those not in T_n) of $T(T_n)$ by u_{ij} where $\pi(v_i) < \pi(u_{ij}) < \pi(v_{i+1})$ for $1 \leqslant i < n$ and, for a given i, $\pi(u_{ij}) < \pi(u_{ij'})$ for $1 \leqslant j < j' \leqslant x_i$. Thus we have denoted the number of extra vertices between v_i and v_{i+1} in the acyclic order π by x_i. Using this notation, the reversing number of T_n is $\sum_{k=1}^{n-1} x_k$.

Recall that the backwards arcs relative to an ordering σ in $T(T_n)$ are arcs $(y, z) \in T(T_n)$ with $\sigma(z) < \sigma(y)$. For any ordering σ of the vertices of $T(T_n)$ the number of backwards arcs relative to σ is at least as large as the number of arcs in T_n, i.e., at least $n(n-1)/2$. This holds since T_n is a minimum reversing set of $T(T_n)$. By Lemma 3 a similar condition holds for certain subtournaments of $T(T_n)$. For any ordering σ of the vertices of $T(T_n)$ restricted to a segment (in the order π) $V'_{jk} = \{v_j, v_{j+1}, \ldots, v_k\} \cup \{u_{rs}: j \leqslant r < k,\ 1 \leqslant s \leqslant x_r\}$, the number of backwards arcs in the segment relative to σ is at least as large as $(k-j+1)(k-j)/2$, the number of arcs in T_n restricted to the segment.

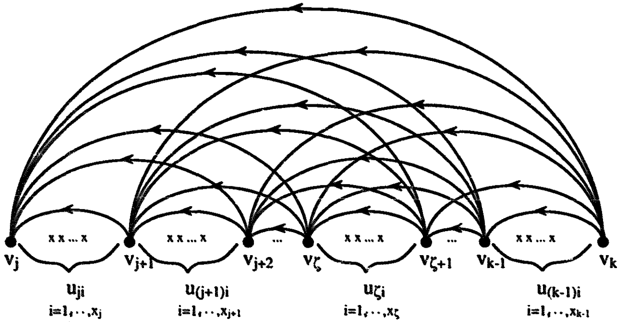
We make use of one "bad" ordering to get a set of inequalities on the x_i which can then be combined to get a lower bound on the reversing number. This ordering applied to the subtournament of $T(T_n)$ induced by V'_{jk} places all the extra vertices u_{rs} to the "right" or "left" (in their natural order consistent with π), and the vertices v_c which appear in T_n in the "middle" in the acyclic order π' of T_n. That is, for a given $j < k$, for $0 \leqslant a, a' < k - 1$, $1 \leqslant b \leqslant x_{j+a}$, $1 \leqslant b' \leqslant x_{j+a'}$ and for $c = j, j+1, \ldots, k$, the ordering σ on V'_{jk} is given by

$$\sigma(u_{ab}) < \sigma(u_{a'b'}) \Leftrightarrow a < a' \text{ or } a = a' \text{ and } b < b',$$

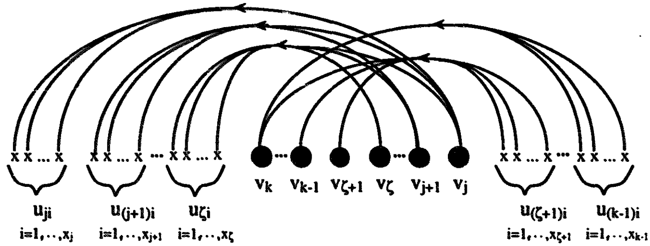
$$\sigma(u_{(j+a)b}) < \sigma(v_c) \Leftrightarrow a \leqslant \left\lfloor \frac{k-j-1}{2} \right\rfloor,$$

$$\sigma(v_c) < \sigma(v_{c'}) \Leftrightarrow c > c'.$$

Fig. 9(a) shows the backwards arcs in the subtournament of T_n on V'_{jk} relative to the ordering π and Fig. 9(b) shows the backwards arcs in the subtournament of T_n on V'_{jk} relative to the ordering σ. From Fig. 9(b) (or from the definitions of T_n, π and σ), it can be checked that the backwards arcs in T_n restricted to V'_{jk} are: for each $0 \leqslant a \leqslant \lfloor (k-j-1)/2 \rfloor$, $(v_c, u_{(j+a)b})$ for $j \leqslant c \leqslant j + a$ and $1 \leqslant b \leqslant x_{j+a}$ and, for each $\lfloor (k-j-1)/2 \rfloor < a < k - j$, $(u_{(j+a)b}, v_c)$ for $j + a + 1 \leqslant c \leqslant k$ and $1 \leqslant b \leqslant x_{j+a}$.



(a) $T(T_n)\big|_{V'_{jk}}$ under the ordering π.



(b) $T(T_n)\big|_{V'_{jk}}$ under the ordering σ.

Fig. 9. Backwards arcs in the subtournament of T_n on V'_{jk} relative to π and σ. (All arcs which are not shown are directed from left to right in the figure.)

Making use of the fact that for each i, there are x_i vertices u_{ij}, we have the following count on the number z of backwards arcs relative to σ. For given k, j, we have

$$
\begin{aligned}
z &= \sum_{a=0}^{\lfloor \frac{k-j-1}{2} \rfloor} \sum_{c=j}^{j+a} \sum_{b=1}^{x_{j+a}} 1 + \sum_{a=\lfloor \frac{k-j-1}{2} \rfloor + 1}^{k-j-1} \sum_{c=j+a+1}^{k} \sum_{b=1}^{x_{j+a}} 1 \\
&= \sum_{a=0}^{\lfloor \frac{k-j-1}{2} \rfloor} (a+1)x_{j+a} + \sum_{a=\lfloor \frac{k-j-1}{2} \rfloor + 1}^{k-j-1} (k-j-a)x_{j+a} \\
&= \sum_{i=1}^{\lfloor \frac{k-j-1}{2} \rfloor + 1} i x_{j+i-1} + \sum_{i=1}^{\lceil \frac{k-j-1}{2} \rceil} i x_{k-i}.
\end{aligned}
$$

In the last line, we have made the change of counters $i = a + 1$ in the first sum and $i = k - j - a$ in the second sum. When $k - j$ is even, both sums have the same number of terms. Combining these we get

$$z = \sum_{i=1}^{\frac{k-j}{2}} i(x_{j+i-1} + x_{k-i}).$$

When $k - j$ is odd, the first sum has one more term than the second. Writing the last term of the first sum separately and combining the remaining terms from both sums, we get

$$z = \left[\sum_{i=1}^{\frac{k-j-1}{2}} i(x_{j+i-1} + x_{k-i})\right] + \frac{k-j+1}{2} x_{j+(k-j-1)/2}.$$

Since the number of backwards arcs relative to σ is at least as large as $(k-j+1)(k-j)/2$, we get the following inequalities:

$$\sum_{i=1}^{\frac{k-j}{2}} i(x_{j+i-1} + x_{k-i}) \geqslant \frac{(k-j+1)(k-j)}{2} \quad \text{for } k-j \text{ even,} \tag{3}$$

$$\left[\sum_{i=1}^{\frac{k-j-1}{2}} i(x_{j+i-1} + x_{k-i})\right] + \frac{k-j+1}{2} x_{j+(k-j-1)/2} \geqslant \frac{(k-j+1)(k-j)}{2} \quad \text{for } k-j \text{ odd,} \tag{4}$$

where the first term in the sum is interpreted as 0 if $k - j = 1$.

At this point, we have inequalities (3) and (4) which provide lower bounds on expressions involving the number of extra vertices x_i. By taking appropriate positive multiples of these inequalities and then summing we can obtain an inequality which provides a lower bound on $\sum_{h=1}^{n-1} x_h$, which is the reversing number. In order to describe the multipliers for the inequalities, we will recursively construct a collection of inequalities (3) and (4) for which the number of copies of each particular inequality will provide the multiplier.

For a given $p = p_0$, we consider the collection $\mathscr{C}_p$ of inequalities defined as follows. Include an inequality for each $0 \leqslant h \leqslant \lfloor \log p \rfloor$. To obtain the hth inequality, define p_h recursively by $p_h = \lfloor p_{(h-1)}/2 \rfloor$. Set $j = 1$ and $k = p_h$. Then use inequality (3) if $k - j$ is even, and $k \neq j$; the empty inequality $0x_1 \geqslant 0$ if $k = j$ and the inequality (4) if $k - j$ is odd, in each case multiplied by 2^h.

For example, with $p = 4$ the inequalities in $\mathscr{C}_4$ are

$$x_1 + 2x_2 + x_3 \geqslant \tfrac{(4)(3)}{2} = 6 \quad (h = 0),$$

$$2(x_1 \geqslant \tfrac{(2)(1)}{2} = 1) \quad (h = 1).$$

(There is no inequality for $h = 2$, since here $p_2 = 1$, and $j = k = 1$.)

Summing the inequalities in $\mathscr{C}_p$ we obtain an inequality of the form

$$\sum_{m=1}^{p-1} c_m x_m \geqslant f(p).$$

We demonstrate by induction the following bounds on the values of the coefficients c_m and the right-hand side $f(p)$.

(a) $c_m \leqslant p - m$.

(b) $f(p) \geqslant p^2 - 2p\log p$.

For $p = 2, 3$ one can easily check that (a) and (b) hold. For $p = 4$, summing the inequalities noted above gives

$$3x_1 + 2x_2 + x_3 \geqslant 8,$$

which satisfies (a) and (b).

Assume that (a) and (b) hold for numbers smaller than p. Given $p \geqslant 5$ the collection $\mathscr{C}_p$ contains one copy of (3) or (4) for $j = 1$ and $k = p$ and for each inequality appearing in $\mathscr{C}_{\lfloor p/2 \rfloor}$ the inequality multiplied by two.

Thus, for $m > \lfloor p/2 \rfloor$, the coefficient c_m is $p - m$ by construction. For $m \leqslant \lfloor p/2 \rfloor$,

$$c_m \leqslant \left[2\left(\left\lfloor \frac{p}{2} \right\rfloor - m\right)\right] + m = 2\left\lfloor \frac{p}{2} \right\rfloor - m \leqslant p - m.$$

Here the term in brackets follows by induction on the inequalities in $\mathscr{C}_{\lfloor p/2 \rfloor}$ which are multiplied by two, and the final m is the coefficient in the new inequality. (Note that in the new inequality, we have $k \neq j$ since $k = p_h$.) This proves that (a) holds for all p.

Now, we show (b). We also have that $f(p) \geqslant 2f(\lfloor p/2 \rfloor) + p(p-1)/2$. The first term follows from the inequalities in $\mathscr{C}_{\lfloor p/2 \rfloor}$ which are multiplied by two, and the final term from the new inequality with $j = 1$ and $k = p$. We now use the inductively assumed bound for $f(\lfloor p/2 \rfloor)$. For p even, $p \geqslant 6$, we get

$$\begin{aligned} f(p) &\geqslant 2\left(\left\lfloor \frac{p}{2} \right\rfloor^2 - 2\left\lfloor \frac{p}{2} \right\rfloor \log \left\lfloor \frac{p}{2} \right\rfloor\right) + \frac{(p)(p-1)}{2} \qquad (5) \\ &= 2\left(\frac{p^2}{4} - p(\log p - 1)\right) + \frac{p^2}{2} - \frac{p}{2} \\ &= p^2 - 2p\log p + \tfrac{3}{2}p \\ &\geqslant p^2 - 2p\log p. \end{aligned}$$

For p odd, $p \geqslant 5$, we get

$$f(p) \geqslant 2\left(\left\lfloor\frac{p}{2}\right\rfloor^2 - 2\left\lfloor\frac{p}{2}\right\rfloor\log\left\lfloor\frac{p}{2}\right\rfloor\right) + \frac{(p)(p-1)}{2} \tag{6}$$

$$= 2\left(\frac{(p-1)^2}{4} - (p-1)(\log(p-1)-1)\right) + \frac{p^2}{2} - \frac{p}{2}$$

$$= p^2 - 2p\log(p-1) + \frac{p}{2} - \frac{3}{2} + 2\log(p-1)$$

$$\geqslant p^2 - 2p\log p.$$

Thus (b) holds.

Similarly to $\mathscr{C}_p$, we can define for a given n, collections $\mathscr{C}'_p$. These include an inequality for each h, $0 \leqslant h \leqslant \lfloor \log p \rfloor$. To obtain the hth inequality, let $p_h = p$ and recursively define $p_h = \lfloor p_{(h-1)}/2 \rfloor$ as before. Set $j = n - p_h + 1$ and $k = n$ and use for the hth inequality (3) if $k - j$ is even and $k \neq j$; the empty inequality $0x_i \geqslant 0$ if $k = j$; and the inequality (4) if $k - j$ is odd.

The sets of inequalities $\mathscr{C}_p$ and $\mathscr{C}'_p$ are symmetric in the sense we now make precise. Consider $\mathscr{C}_p$ when $j = 1$ and $k = p_h$ and $\mathscr{C}'_p$ when $j = n - p_h + 1$ and $k = n$. Then $k - j$ is $p_h - 1$ in both cases, so we use the same inequality (3) or (4) in each case. Whenever in (3) or (4) in $\mathscr{C}_p$ there is a term $ix_i = ix_{j+i-1}$, then in (3) or (4) in $\mathscr{C}'_p$ there is a corresponding term $ix_{k-i} = ix_{n-i}$. Whenever in (3) or (4) in $\mathscr{C}_p$ there is a term $ix_{k-i} = ix_{p_h-i}$, then in (3) or (4) in $\mathscr{C}'_p$ there is a corresponding term $ix_{j+i-1} = ix_{n-(p_h-i)}$. Whenever in (4) in $\mathscr{C}_p$ there is a term

$$\frac{k-j+1}{2}x_{(j+(k-j+1)/2)} = \frac{p_h - 1 + 1}{2}x_{1+(p_h-1+1)/2} = \frac{p_h}{2}x_{p_h/2},$$

then in (4) in $\mathscr{C}'_p$ there is a corresponding term

$$\frac{k-j+1}{2}x_{(j+(k-j+1)/2)} = \frac{n-(n-p_h+1)+1}{2}x_{n-p_h+1+(n-(n-p_h+1)-1)/2} = \frac{p_h}{2}x_{(n-p_h)/2}.$$

In all cases, whenever there is a term x_m in the set of inequalities $\mathscr{C}_p$, there is a corresponding term x_{n-m} with the same coefficient in the set of inequalities $\mathscr{C}'_p$.

As with $\mathscr{C}_p$, summing the inequalities in $\mathscr{C}'_p$ we obtain an inequality of the form

$$\sum_{m=n-p+1}^{n-1} c'_m x_m \geqslant f'(p),$$

where

(a') $c'_{n-m} \leqslant p - m$,

(b') $f'(p) \geqslant p^2 - 2p\log p$.

By the symmetry to $\mathscr{C}_p$, with x_{n-m} replacing x_m, (a') and (b') hold.

Finally to get a bound on $\sum_{i=1}^{n-1} x_i$ we use the following collection of inequalities:
(i) One copy of inequality (3) or (4) for $j = 1$ and $k = n$.
(ii) One copy of the collection $\mathscr{C}_{\lfloor n/2 \rfloor}$.
(iii) One copy of the collection $\mathscr{C}'_{\lfloor n/2 \rfloor}$.

Summing inequalities from (i), (ii), and (iii) we get an inequality

$$\sum_{m=1}^{n-1} d_m x_m \geqslant \frac{(n)(n-1)}{2} + f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f'\left(\left\lfloor \frac{n}{2} \right\rfloor\right). \tag{7}$$

The right-hand side of this inequality is the sum of the bounds for (i), (ii), and (iii).

For the coefficients d_m on the left side of the inequality, note that in $\mathscr{C}_{\lfloor n/2 \rfloor}$ the only nonzero coefficients are for $\{x_1, \ldots, x_{\lfloor n/2 \rfloor - 1}\}$ and in $\mathscr{C}'_{\lfloor n/2 \rfloor}$ the only nonzero coefficients are for $\{x_{n - \lfloor n/2 \rfloor + 1}, \ldots, x_{n-1}\}$. Note that $n - \lfloor n/2 \rfloor + 1 = \lceil n/2 \rceil + 1$ so the nonzero coefficients from (ii) and (iii) do not overlap. Consider the coefficient d_m for $m < \lfloor n/2 \rfloor$. In this case, $d_m \leqslant (\lfloor n/2 \rfloor - m) + m \leqslant n/2$. Here the first term is the coefficient from (ii) with the bound (a) and the final m is the coefficient of x_m in (i). For d_m if $m > \lceil n/2 \rceil$, we get the same bound from (iii) and (a′) and (i). When n is even $x_{n/2}$ appears only in (i) and has coefficient $n/2$. For n odd, $x_{\lfloor n/2 \rfloor}$ and $x_{\lceil n/2 \rceil}$ appear only in (i) with coefficient $\lfloor n/2 \rfloor$. So the coefficients d_m are all less than or equal to $n/2$.

Also note that substituting the bounds (b) and (b′) for $f(\lfloor n/2 \rfloor)$ and $f'(\lfloor n/2 \rfloor)$ into the right-hand side of (7) we get the same right-hand side as in (5) and (6) with n instead of p. Thus, as in (5) and (6), we get the right-hand side of (7) greater than or equal to $n^2 - 2n \log n$. Using this bound and the bound $d_m \leqslant n/2$, we get from (7) that

$$\frac{n}{2} \sum_{m=1}^{n-1} x_m \geqslant \sum_{m=1}^{n-1} d_m x_m \geqslant \frac{(n)(n-1)}{2} + f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f'\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \geqslant n^2 - 2n \log n.$$

Hence,

$$\sum_{m=1}^{n-1} x_m \geqslant \frac{2}{n}(n^2 - 2n \log n) = 2n - 4 \log n,$$

giving the desired lower bound on the reversing number of T_n.

For the upper bound we construct a tournament T on $3n - 4$ vertices with T_n as a minimum reversing set when $n \geqslant 4$. Let T_n have acyclic ordering $\pi(v_n) < \pi(v_{n-1}) < \cdots < \pi(v_1)$. Let T' be an acyclic tournament with vertex set $V(T') = V(T_n) \cup \{u_{11}, u_{(n-1)0}\} \cup \{u_{ij}\colon 2 \leqslant i \leqslant n-2, j = 0, 1\}$ and acyclic ordering π' satisfying $\pi'(v_1) < \pi'(u_{11}) < \pi'(v_2)$, $\pi'(v_i) < \pi'(u_{i0}) < \pi'(u_{i1}) < \pi'(v_{i+1})$ for $2 \leqslant i \leqslant n-2$, and $\pi'(v_{n-1}) < \pi'(u_{(n-1)0}) < \pi'(v_n)$. Since $T_n^{\mathrm{R}} \subset T'$, we can define $T = (T' \setminus T_n^{\mathrm{R}}) \cup T_n$. T_n is shown in Fig. 10. By the construction of T, T_n is a reversing set of T. To show that T_n is a minimum reversing set of T we consider the following set τ of $n(n-1)/2$ triples:

$$\tau = \tau_1 \cup \tau_2 \cup \tau_3,$$

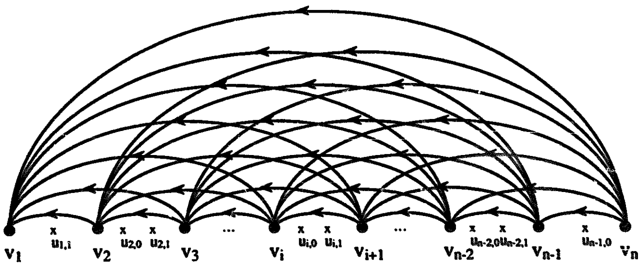


Fig. 10. T with T_n as a minimum reversing set. (All arcs which are not shown are directed from left to right in the figure.)

where

$$\tau_1 = \{(v_i, u_{(k_{ij}, \delta_{ij})}, v_j)\colon 1 \leqslant i < j \leqslant n-1\},\ \tau_2 = \{(v_1, u_{(k_{1n}, 1)}, v_n)\},$$

$$\tau_3 = \{(v_i, u_{i0}, v_n)\colon 2 \leqslant i \leqslant n-1\}$$

with

$$k_{ij} = \left\lfloor \frac{j+i}{2} \right\rfloor \quad \text{and} \quad \delta_{ij} = (j-i) \bmod 2.$$

(Notice that, if $n \geqslant 4$, these triples are indeed constructed with $2n-4$ "extra" vertices, i.e., we need no u_{10} or $u_{(n-1)1}$ to build them.)

It is easy to check that the orientation of the arcs of these triples is such that every one of them is in fact a 3-cycle. So it is enough to verify that these $n(n-1)/2$ 3-cycles are arc disjoint to complete the proof. First, notice that if we have $k_{ij}\delta_{ij} = k_{rs}\delta_{rs}$ and $i = r$ or $j = s$, then $(i, j) = (r, s)$. So, if two 3-cycles from $\tau_1 \cup \tau_2$ have a common arc (two common vertices), then it is the same 3-cycle. Therefore, the 3-cycles from $\tau_1 \cup \tau_2$ are arc disjoint. On the other hand, the 3-cycles from $\tau_2 \cup \tau_3$ are obviously arc disjoint. Finally, consider a 3-cycle from τ_1: $(v_i, u_{(k_{ij}, \delta_{ij})}, v_j)$ with $1 \leqslant i < j \leqslant n-1$, and a 3-cycle from τ_3: (v_r, u_{r0}, v_n) with $2 \leqslant r \leqslant n-1$. If they had a common arc, it would necessarily be the arc (v_r, u_{r0}), and then we should have $i = r$ and $(v_i, u_{(k_{ij}, \delta_{ij})}) = (v_i, u_{i0})$. But this equality is not possible, since $k_{ij} = i$ and $j > i$ imply $j = i+1$, and so $\delta_{ij} = 1$. □

We note at this point that we could set up an integer linear program to minimize the sum of the x_i subject to inequality (3) or (4) for all j and k with $1 \leqslant j < k \leqslant n$. The solution of this would provide a bound on the reversing number. It would be interesting to see if the bound derived from this integer program is tight. The multipliers used in the collection of inequalities used in the proof of the lower bound can be viewed as variables in a dual feasible solution to the linear program obtained by relaxing the integer constraints. Notice that the upper bound $2n-4$ is not tight in all cases, as can be seen in Table 1, which lists exact values of $r(T_n)$ for small n. The values in this table have been calculated by special cases of the techniques in the proof.

Table 1
Exact values of $r(T_n)$ for small n

n	2	3	4	5	6	7	8	9	10	11	12
$r(T_n)$	1	3	4	6	8	10	11	14	15	17	19

6. Reversing numbers of acyclic digraphs in some special classes

In this section we compute the reversing number for acyclic digraphs in various special classes.

6.1. Stars

Let a *directed star* S_n be a digraph on n vertices with a distinguished vertex v such that all arcs in S_n contain v as either head or tail. Note that S_n contains $n-1$ arcs and by our convention of denoting by $|S_n|$ the size of the arc set of S_n, we have $|S_n| = n-1$.

Theorem 19. *If S_n is a directed star on n vertices then $r(S_n) = n-1$.*

Proof. By Lemmas 10 and 12, we may assume that S_n is the directed star in which $v = v_0$ is the head of all arcs, i.e., $S_n = \{(v_i, v_0): i = 1,2,\dots,n-1\}$. Let T realize S_n and let π be the acyclic ordering of $(T\backslash S_n)\cup S_n^R$. Since $(v_0, v_i) \in S_n^R$, $\pi(v_0) < \pi(v_i)$, $i = 1,2,\dots,n-1$. Without loss of generality, $\pi(v_0) < \pi(v_1) < \pi(v_2) < \cdots < \pi(v_{n-1})$. Also, by Lemma 2, we may assume that there are no "extra" vertices w, i.e., vertices in $V(T)\backslash V(S_n)$, such that $\pi(w) > \pi(v_{n-1})$ or $\pi(w) < \pi(v_0)$. For $i = 1,2,\dots,n-1$, let there be k_i extra vertices $x_{i1}, x_{i2}, \dots, x_{ik_i}$ between v_{i-1} and v_i in π, i.e., $\pi(v_{i-1}) < \pi(x_{ij}) < \pi(v_i)$, for $j = 1,2,\dots,k_i$.

Note that $\sum_{i=1}^{n-1} k_i = r(S_n)$. Let $X = \{(v_0, x_{ij}): i = 1,2,\dots,n-1,\ j = 1,2,\dots,k_i\}$. Then $X \subseteq T$ and $(T\backslash X)\cup X^R$ is acyclic, with the acyclic order π' obtained from π by making v_0 a sink instead of a source and maintaining the acyclic order among the other vertices. That is, $\pi'(u) = \pi(u) - 1$ for $u \neq v_0$ and $\pi'(v_0) > \pi'(v_{n-1}) > \pi'(u)$ for all $u \in V(T)$. Since S_n is a minimum reversing set of T,

$$|X| \geqslant |S_n| = n-1.$$

Note that $|X| = \sum_{i=1}^{n-1} k_i = r(S_n)$. Therefore, $r(S_n) \geqslant n-1$. Letting $k_i = 1$ for all i gives a tournament of $2n-1$ vertices containing the $n-1$ arc disjoint 3-cycles (x_{i1}, v_i, v_0), $i = 1,\dots,n-1$, with S_n as a reversing set and thus a minimum reversing set by Lemma 5. □

6.2. Disjoint arcs

As mentioned above, there exist digraphs whose reversing number is 0. An example will be the disjoint union of n arcs, the graph we denote by E_n.

Theorem 20. *$r(E_1) = r(E_2) = 1$, and $r(E_n) = 0$, $n \geqslant 3$.*

Proof. Note that $E_1 = P_2$. Therefore, by Theorem 13, $r(E_1) = 1$.

By Theorem 17, $r(E_2) > 0$ since E_2 has only 4 vertices. Let T' be given by the digraph in Fig. 11. E_2 is clearly a reversing set of T'. Also the two arc disjoint 3-cycles (v_3, v_1, x) and (v_4, v_2, v_3) imply that the reversal of one arc of T' will not produce an acyclic tournament. Therefore, T' realizes E_2 and $r(E_2) = 1$.

Let $n \geqslant 3$ and let the E_n be defined by

$$V(E_n) = \{v_1, v_2, \ldots, v_{2n}\},$$

$$A(E_n) = \{(v_{n+1}, v_1), (v_{n+2}, v_2), \ldots, (v_{2n}, v_n)\}.$$

Let T' be the acyclic tournament on $V(E_n)$ with acyclic ordering π such that $\pi(v_i) = i$. Note that $E_n^{\mathrm{R}} \subseteq T'$. Let $T = (T' \setminus E_n^{\mathrm{R}}) \cup E_n$. Hence, E_n is a reversing set of T. Next, we will exhibit n arc disjoint 3-cycles in T. Since there are n arcs in E_n, this will imply by Lemma 5 that E_n is a minimum reversing set of T, i.e., T realizes E_n. Therefore, since $|V(T)| = |V(E_n)|$, $r(E_n) = 0$.

Let

$$\tau = \{(v_1, v_2, v_{n+1}), (v_2, v_3, v_{n+2}), \ldots, (v_{n-1}, v_n, v_{2n-1})\} \cup \{(v_n, v_{2n-2}, v_{2n})\}.$$

It is an easy exercise to see that τ contains n arc disjoint 3-cycles from T, provided that $n \geqslant 3$. $\square$

6.3. Complete bipartite digraphs

In this section we compute $r(K_{m,n})$, where

$$V(K_{m,n}) = \{v_1, v_2, \ldots, v_m\} \cup \{u_1, u_2, \ldots, u_n\}$$

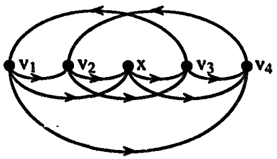


Fig. 11. T' realizing E_2.

and

$$A(K_{m,n}) = \{(v_i, u_j): i = 1, \ldots, m, j = 1, \ldots, n\}.$$

$K_{m,n}$ will be called a *complete bipartite* digraph.

We will make use of Latin rectangles in the next proof. An $m \times n$ *Latin rectangle* with entries from a set S of n distinct elements is an array with entries from S such that no element of S appears twice in the same row or in the same column. It is not difficult to show, using for example Hall's marriage theorem, that $m \times n$ Latin rectangles exist for $m = 1, \ldots, n$ (see for example [30]).

Theorem 21. $r(K_{m,n}) = \max\{m, n\}$.

Proof. By Lemma 10, we may assume that $\max\{m, n\} = m$.

First we show that $r(K_{m,n}) \leqslant m$. Let T' denote the acyclic tournament on $V(K_{m,n}) \cup \{x_1, x_2, \ldots, x_m\}$ with acyclic ordering π such that

$$\pi(u_i) = i, \quad i = 1, 2, \ldots, n,$$

$$\pi(x_j) = n + j, \quad j = 1, 2, \ldots, m,$$

$$\pi(v_k) = n + m + k, \quad k = 1, 2, \ldots, m.$$

Note that $K^{\mathrm{R}}_{m,n} \subseteq T'$. Let $T = (T' \setminus K^{\mathrm{R}}_{m,n}) \cup K_{m,n}$. Hence, $K_{m,n}$ is a reversing set of T.

Since there are mn arcs in $K_{m,n}$, if we can exhibit mn arc disjoint cycles in T, this will imply by Lemma 5 that $K_{m,n}$ is a minimum reversing set of T and hence $r(K_{m,n}) \leqslant m$. Let L be an $m \times n$ Latin rectangle with entries from $x_1, x_2, \ldots, x_m$. Consider the mn 3-cycles (u_j, L_{ij}, v_i) for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. Since L is a Latin rectangle, $i \neq i' \Rightarrow L_{ij} \neq L_{i'j}$ and $j \neq j' \Rightarrow L_{ij} \neq L_{ij'}$. Thus the mn 3-cycles are arc disjoint.

Next, suppose that $r(K_{m,n}) < m$. Therefore, there exists a tournament T with minimum reversing set $K_{m,n}$ such that $|V(T)| < m + n + m$. Without loss of generality, we may assume that the acyclic ordering π' of the vertices of $T' = (T \setminus K_{m,n}) \cup K^{\mathrm{R}}_{m,n}$ satisfies

$$\pi'(u_1) < \pi'(u_2) < \cdots < \pi'(u_n) < \pi'(v_1) < \pi'(v_2) < \cdots < \pi'(v_m).$$

Let $\{x_1, x_2, \ldots, x_k\}$ be the extra vertices in T, i.e., $\{x_1, x_2, \ldots, x_k\} = V(T) \setminus V(K_{m,n})$, and note that $k < m$. Also note that every directed cycle in T must contain an arc of the form (u_i, x_j) where $\pi'(u_i) < \pi'(x_j)$. Let $X = \{(u_i, x_j): \pi'(u_i) < \pi'(x_j)\} \subseteq T$. Thus X is a transversal of the cycles and by the remarks in the introduction, the minimum size of a transversal is equal to the size of a minimum reversing set. Thus the size of a minimum reversing set of T is at most $|X| \leqslant kn < mn = |K_{m,n}|$. This contradicts the assumption that $K_{m,n}$ is a minimum reversing set of T. Therefore $r(K_{m,n}) \geqslant m$. Combining the two inequalities we have $r(K_{m,n}) = m$. $\square$

Notice that the result for complete bipartite digraphs yields an alternative proof of the result on stars since $K_{1,n-1}$ is a directed star.

6.4. Alternating paths

We have shown in the case $n \geqslant 8$ of the proof of Theorem 17 that the reversing number of alternating paths on eight or more vertices is 0. We now determine the reversing number of all alternating paths.

Theorem 22. *Let A_n be an alternating path on n vertices. Then,*

$$r(A_n) = \begin{cases} 1 & \text{if } n = 2,4,5,6,7, \\ 2 & \text{if } n = 3, \\ 0 & \text{if } n \geqslant 8. \end{cases}$$

Proof. As noted in the proof of Theorem 17, Lemma 10 says that $r(D) = r(D^R)$ for all D. Thus, we may assume that A_n is labeled with vertex set $\{v_1, \ldots, v_n\}$ and arc set $\{(v_i, v_{i+1}), (v_i, v_{i-1})$: i is odd, and both vertices are in $V\}$. The cases $n \geqslant 8$ were shown in the proof of the case $n \geqslant 8$ of Theorem 17. Thus we must consider the cases $n \leqslant 7$.

Case $n = 2,3$: Note that A_2 and A_3 are directed stars on two and three vertices, respectively. Thus, by Theorem 19, $r(A_2) = 1$ and $r(A_3) = 2$.

Case $n = 4,5,6$: By Theorem 17, $r(A_4), r(A_5), r(A_6) > 0$. Fig. 12 shows directed tournaments $T'(A_4)$, $T'(A_5)$, and $T'(A_6)$ on 5, 6, and 7 vertices, respectively, which can easily be shown to have reversing sets A_4, A_5, and A_6, respectively. Also, in Fig. 12 we list 3, 4, and 5 arc disjoint cycles from $T'(A_4)$, $T'(A_5)$, and $T'(A_6)$, respectively, to show that $T'(A_4), T'(A_5), T'(A_6)$ realize A_4, A_5, A_6, respectively. Thus $r(A_4) = r(A_5) = r(A_6) = 1$.

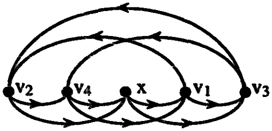
Case $n = 7$: We show that $r(A_7) \leqslant 1$, by the tournament in Fig. 13.

Next we must show that A_7 is not a minimum reversing set of any tournament on 7 vertices. Suppose that there exists a tournament T^* on 7 vertices with A_7 as a minimum reversing set.

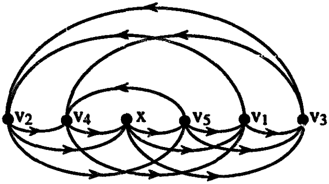
We first show that the outdegrees of T^* must be in $\{2,3,4\}$. If there were a vertex x in T^* with $d^+_{T^*}(x) = 5$ or 6 (respectively 0 or 1), then by reversing at most one arc, a tournament T with x as a source (respectively sink) is obtained. Recall the result of Bermond and Kodratoff [6], used in Theorem 17, that m_6, the size of a largest minimum reversing set for a tournament on 6 vertices, is 4. Then $T|_{V(T^*)\setminus x}$ can be made acyclic with at most four reversals and, by Lemma 2, the size of a minimum reversing set of T^* is at most five. Thus all outdegrees in T^* must be 2, 3 or 4.

The outdegrees in T^* cannot all be 3, since in any reversing set the vertex which becomes the sink after reversal must be contained in three arcs which are reversed and there is no such vertex in A_7.

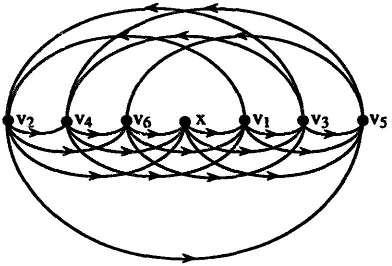
Thus, since the sum of the outdegrees of vertices in T^* is $n(n-1)/2 = 21$, the multiset of outdegrees for T^* must be one of $\{2,3,3,3,3,3,4\}$, $\{2,2,3,3,3,4,4\}$, or $\{2,2,2,3,4,4,4\}$. The outdegrees after reversal of the arcs in a minimum reversing set are $\{0,1,2,3,4,5,6\}$. Since the arcs of A_7 are those which are reversed in T^* to make the tournament acyclic, we see that the changes in outdegrees from T^* to



(a) $T'(A_4)$ containing arc disjoint cycles (v_1, v_2, v_4), (v_3, v_2, x, v_1) and (v_3, v_4, x).



(b) $T'(A_5)$ containing arc disjoint cycles (v_1, v_2, x), (v_3, v_2, v_5), (v_3, v_4, v_1) and (v_5, v_4, x).



(c) $T'(A_6)$ containing arc disjoint cycles (v_1, v_2, v_6), (v_3, v_2, x) (v_3, v_4, v_1), (v_5, v_4, x) and (v_5, v_6, v_3).

Fig. 12. Tournaments realizing alternating paths A_4, A_5, A_6.

$(T^* \setminus A_7) \cup A_7^R$ must be exactly three increases by two, two decreases by two, and two decreases by one. It is easy to see that these changes cannot transform the outdegrees $\{2,3,3,3,3,3,4\}$ into $\{0,1,2,3,4,5,6\}$. Thus $\{2,3,3,3,3,3,4\}$ cannot be the multiset of outdegrees.

Consider next the case of $\{2,2,2,3,4,4,4\}$. Every tournament contains a Hamiltonian path (see for example [14]). Applying this observation to the subtournament of T^* induced by vertices of outdegree 4, we see that we can find x,y,z with $(x,y),(y,z) \in T^*$ and $d^+_{T^*}(x) = d^+_{T^*}(y) = d^+_{T^*}(z) = 4$. Consider an acyclic tournament in

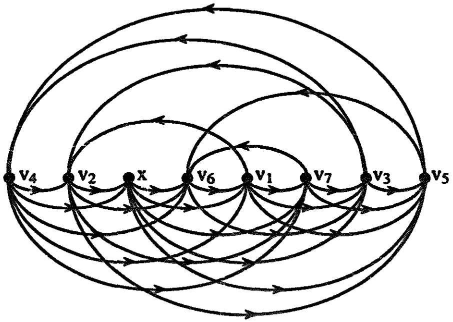


Fig. 13. Tournament with A_7 as a minimum reversing set containing arc disjoint cycles (v_1, v_2, x), (v_3, v_2, v_7), (v_3, v_4, x), (v_5, v_4, v_2), (v_5, v_6, v_3) and (v_7, v_6, v_1).

which x is a source, y is beaten only by x, and z is beaten only by x and y. That is, the acyclic order for T has $\pi(x) < \pi(y) < \pi(z)$ and $\pi(z) < \pi(v)$ for all $v \neq x, y$. Here, two reversals in T^* are needed to make x a source. In T^*, y was beaten by two vertices, one of which was x, so one reversal is needed to put y in order. Also z was beaten by two vertices, y and another (possibly x), so at most one more reversal is needed to place z third in the acyclic order. Finally the remaining vertices form a tournament on four vertices; since $m_4 = 1$ at most one additional reversal is needed to make these acyclic. Thus an acyclic tournament T can be always obtained from T^* with at most five reversals, two for x, one for y, at most one for z, and at most one for the remaining vertices. Hence A_7 cannot be a minimum reversing set of a tournament with outdegrees $\{2,2,2,3,4,4,4\}$.

Finally, consider the outdegrees $\{2,2,3,3,3,4,4\}$. Denote by $X = \{x_1, x_2\}$ the vertices with outdegree four, $Y = \{y_1, y_2, y_3\}$ the vertices with outdegree three, and $Z = \{z_1, z_2\}$ the vertices with outdegree two. Since each vertex is contained in a total of six arcs, both x_1 and x_2 are the heads of two arcs. Assume without loss of generality that x_1 beats x_2, i.e., $(x_1, x_2) \in T^*$.

Consider first the case that there exists a vertex in Y (with outdegree three) which is beaten by both x_1 and x_2. Without loss of generality assume that this vertex is y_1. Since $d^+_{T^*}(y_1) = 3$ and y_1 is beaten by both x_1 and x_2, y_1 must beat three of the four vertices $\{y_2, y_3, z_1, z_2\}$. The acyclic order with $\pi(x_1) < \pi(x_2) < \pi(y_1)$ and $\pi(y_1) < \pi(v)$ for all $v \in \{y_2, y_3, z_1, z_2\}$ can be obtained from T^* as follows: Two reversals for the arcs with x_1 as head, one reversal for the arc other than (x_1, x_2) with x_2 as head, one reversal for the arc from the one vertex in $\{y_2, y_3, z_1, z_2\}$ beating y_1, at most one reversal to put the four vertices $\{y_2, y_3, z_1, z_2\}$ in acyclic order. The last point follows since $m_4 = 1$. Thus an acyclic order is obtained from T^* with a total of at most five reversals, showing that A_7 is not a minimum reversing set of such a tournament.

Otherwise, there is no vertex in Y beaten by both x_1 and x_2. If this is the case, $(x_i, z_j) \in T^*$ for $i, j = 1, 2$. This follows since x_1 beats three of the vertices and x_2 beats four of the vertices among $Y \cup Z$ and no vertex in Y is beaten by both x_1 and x_2. Then x_2 must beat two of the vertices in Y and x_1 must beat one vertex in Y and these must be distinct. So we may assume that $(x_1, y_1), (x_2, y_2), (x_2, y_3) \in T^*$ (and that $(y_1, x_2), (y_2, x_1), (y_3, x_1) \in T^*$). Then T^* is as shown in Fig. 14. Consider the acyclic order with $\pi(x_1) < \pi(y_1) < \pi(x_2)$ and $\pi(x_2) < \pi(v)$ for $v \in \{y_2, y_3, z_1, z_2\}$. This is obtained from T^* by at most five reversals; two for reversing (y_2, x_1) and (y_3, x_1), two for the two arcs from a vertex in $\{y_2, y_3, z_1, z_2\}$ with y_1 as head (since $d^+_{T^*}(y_1) = 3$), and at most one to put $\{y_2, y_3, z_1, z_2\}$ in acyclic order. The last point follows since $m_4 = 1$. Thus A_7 is not a minimum reversing set of T^* in this case, completing the proof that T^* cannot have outdegrees $\{2, 2, 3, 3, 3, 4, 4\}$. This completes the proof that $r(A_7) \neq 0$. □

6.5. Alternating cycles

Let AC_{2n} be the alternating cycle on $2n$ vertices ($n \geqslant 2$), that is, the graph for which there exists a numbering such that $X_{2n} = \{x_i \mid 1 \leqslant i \leqslant 2n\}$ is the set of vertices and $A_{2n} = \{(x_{2i-1}, x_{2i}) \mid 1 \leqslant i \leqslant n\} \cup \{(x_{2i+1}, x_{2i}) \mid 1 \leqslant i \leqslant n-1\} \cup \{(x_1, x_{2n})\}$ is the set of arcs. We now prove the following theorem.

Theorem 23. *Let AC_{2n} be the alternating cycle on $2n$ vertices ($n \geqslant 2$). Then we have: $r(AC_4) = r(AC_6) = 2$; $r(AC_8) = 1$; and $r(AC_{2n}) = 0$ for $n \geqslant 5$.*

Proof. *Case $n = 2$:* Note that AC_4 is the complete bipartite digraph $K_{2,2}$. Thus, by Theorem 21, we have $r(AC_4) = 2$.

Case $n = 3$: Note that an alternating path AP_6 on six vertices is a subgraph (on the same vertex set) of AC_6. Thus, by Theorems 8 and 22, we have $r(AC_6) \geqslant 1$.

Suppose that $r(AC_6) = 1$. Then there exists a tournament T^* on seven vertices with AC_6 as a minimum reversing set. We first show that the outdegrees of T^* must be in

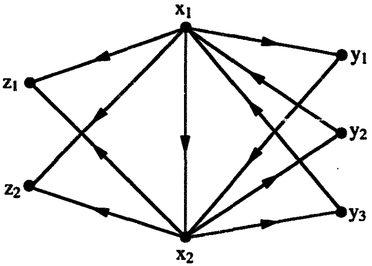


Fig. 14. T^*. (All arcs which are not shown may have any orientation.)

$\{2,3,4\}$. If there were a vertex x in T^* with $d^+_{T^*}(x) = 5$ or 6 (respectively 0 or 1), then by reversing at most one arc, tournament T with x as a source (respectively sink) is obtained. Recall the result of Bermond and Kodratoff [6], mentioned before the proof of Theorem 17, that m_6, the size of a largest minimum reversing set for a tournament on 6 vertices, is 4. Then $T|_{V(T^*)\setminus x}$ can be made acyclic with at most four reversals and, by Lemma 2, the size of a minimum reversing set of T^* is at most five. Thus all outdegrees in T^* must be 2,3 or 4.

Thus, since the sum of the outdegrees of vertices in T^* is $n(n-1)/2 = 21$, the multiset of outdegrees for T^* must be one of $\{3,3,3,3,3,3,3\}$, $\{2,3,3,3,3,3,4\}$, $\{2,2,3,3,3,4,4\}$, or $\{2,2,2,3,4,4,4\}$. The outdegrees after reversal of the arcs in a minimum reversing set are $\{0,1,2,3,4,5,6\}$. Since the arcs of AC_6 are those which are reversed in T^* to make the tournament acyclic, we see that the changes in outdegrees from T^* to $(T^*\setminus AC_6)\cup AC_6^R$ must be exactly three increases by two, three decreases by two, and one vertex with no change (corresponding to the "extra" vertex). It is not difficult to check that of the four possible multisets, only $\{2,2,3,3,3,4,4\}$ can attain $\{0,1,2,3,4,5,6\}$ by these reversals. Thus, we consider this case.

In order to transform $\{2,2,3,3,3,4,4\}$ into $\{0,1,2,3,4,5,6\}$ by the reversal described above it is necessary that the outdegree of the extra vertex is three. For $i = 0,1,\ldots,6$, let v_i denote the vertex with outdegree i in the tournament $(T^*\setminus AC_6)\cup AC_6^R$. Note that v_3 is the extra vertex and that v_0, v_1, and v_2 are the sources in AC_6 and v_4, v_5, and v_6 are the sinks in AC_6. Note also that (v_6,v_5), (v_6,v_4), and (v_6,v_3) are all arcs of T^* since they are arcs of $(T^*\setminus AC_6)\cup AC_6^R$ and not arcs of AC_6. Then, since $d_{T^*}(v_6) = 4$ (its outdegree was increased by two), it must be that $(v_6,v_j)\in T^*$ for exactly one of $j=0$, $j=1$, or $j=2$. We consider each of these possibilities separately. In each case we exhibit a reversing set of size five, contradicting the assumption that T^* has AC_6 as a minimum reversing set.

If $(v_6,v_2)\in T^*$, then (v_2,v_5) and (v_2,v_4) are both in AC_6 and thus in T^*. Consider the acyclic ordering $\pi(v_6)<\pi(v_2)<\pi(v_5)<\pi(v_4)<\pi(v_3)<\pi(v_1)<\pi(v_0)$. This is obtained by reversing the same four arcs on $V(T^*)\setminus\{v_2\}$ as those on AC_6 (which is the subgraph induced on AC_6 by these vertices) and the arc (v_3,v_2), a total of five arcs.

If $(v_6,v_1)\in T^*$, then (v_1,v_5) and (v_1,v_4) are both in AC_6 and thus in T^*. Also, $(v_2,v_6)\in T^*$ by assumption and exactly one of (v_5,v_2) and (v_4,v_2) is in T^*. Consider the acyclic ordering $\pi(v_2)<\pi(v_6)<\pi(v_1)<\pi(v_5)<\pi(v_4)<\pi(v_3)<\pi(v_0)$. This is obtained by reversing the two arcs on the subgraph of AC_6 with v_0 as tail, the arcs (v_3,v_2) and (v_3,v_1), and the arc from $\{(v_5,v_2),(v_4,v_2)\}$ that is in T^*, a total of five arcs.

If $(v_6,v_0)\in T^*$, then (v_0,v_5) and (v_0,v_4) are both in AC_6 and thus in T^*. Also, for $j=1,2$, $(v_j,v_6)\in T^*$ by assumption and exactly one of (v_5,v_j) and (v_4,v_j) is in T^*. Consider the acyclic ordering $\pi(v_2)<\pi(v_1)<\pi(v_6)<\pi(v_0)<\pi(v_5)<\pi(v_4)<\pi(v_3)$. This is obtained by reversing the arcs (v_3,v_i) for $i=1,2,3$ and for $j=1,2$, the arc from $\{(v_5,v_j),(v_4,v_j)\}$ that is in T^*, a total of five arcs.

To complete the proof of the case $n=3$, we exhibit in Fig. 15 a tournament T on eight vertices with AC_6 as a minimum reversing set, showing that $r(AC_6)\leqslant 2$.

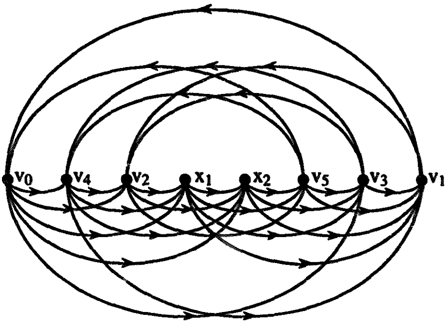


Fig. 15. T on eight vertices with AC_6 as a minimum reversing set, containing arc disjoint cycles (v_0, x_1, v_5), (v_4, x_1, v_3), (v_2, x_1, v_1), (v_0, x_2, v_1), (v_4, x_2, v_5) and (v_2, x_2, v_3).

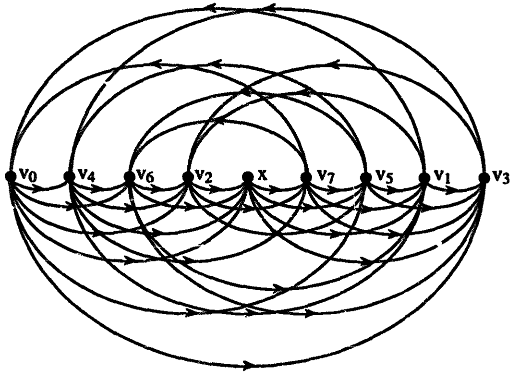


Fig. 16. T on nine vertices with AC_8 as a minimum reversing set, containing arc disjoint cycles (v_0, v_4, v_7), (v_0, x, v_1), (v_4, x, v_5), (v_4, v_1, v_3), (v_6, x, v_7) (v_6, v_2, v_5) (v_2, x, v_3) and (v_2, v_7, v_1).

Case n = 4: We first show that $r(AC_8) \neq 0$. Let x be the vertex which is the source in the acyclic order of the tournament obtained by reversing the arcs of AC_8 in a tournament T with $V(T) = V(AC_8)$ that realizes AC_8. By Lemma 3, the alternating path on seven vertices obtained by deleting the vertex x from AC_8 is a minimum reversing set of the tournament T restricted to $V(T)\backslash\{x\} = V(AC_8)\backslash\{x\}$. This would be a tournament on seven vertices with an alternating path on seven vertices as

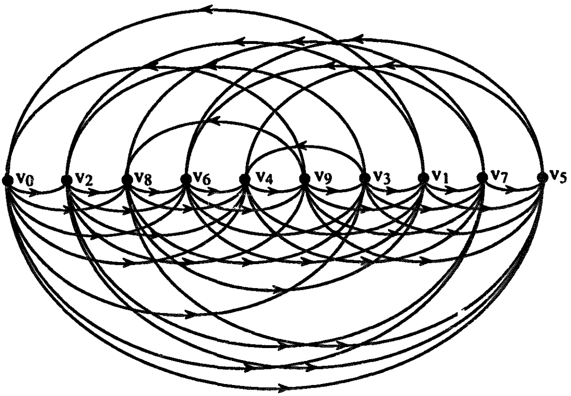


Fig. 17. T on ten vertices with AC_{10} as a minimum reversing set, containing arc disjoint cycles (v_0, v_2, v_9), (v_0, v_3, v_1), (v_2, v_8, v_3), (v_2, v_6, v_1), (v_8, v_1, v_7), (v_8, v_6, v_9) (v_6, v_4, v_7) (v_6, v_3, v_5) (v_4, v_9, v_3) and (v_4, v_1, v_5).

a minimum reversing set, contradicting $r(A_7) = 1$, which was shown in Theorem 22. Thus, $r(AC_8) \geqslant 1$ and in Fig. 16 we exhibit a tournament on nine vertices with AC_8 as a minimum reversing set. So $r(AC_8) = 1$.

Case $n = 5$: Fig. 17 exhibits a tournament on 10 vertices with AC_{10} as a minimum reversing set, showing that $r(AC_{10}) = 0$.

Case $n \geqslant 6$: Let $m = 2n - 1$. Consider $T(A_m)$ and τ_m as constructed in the proof of the case $n \geqslant 8$ of Theorem 17. By conditions (e) and (b) of that proof, (v_m, v_{m-1}) is the only arc in a cycle of τ_m containing v_m and $(v_2, v_m) \in T(A_m)$. It can easily be seen that the extension from A_9 to A_{10} can be done so that $(v_{10}, v_1) \in T(A_{10})$ is not an arc of any cycle of τ_{10} (see condition (e) of the proof). Then, $(v_{10}, v_1) \in T(A_m)$ for $m \geqslant 10$ since $T(A_{10})$ is a subtournament of the tournaments constructed by the extensions. Additionally, since (v_{10}, v_1) is not an arc of a cycle in τ_{10}, it is not an arc of a cycle in τ_m ($m \geqslant 10$) by the construction of the τ extensions. Let $S = \{(v_{10}, v_1), (v_2, v_m)\}$. Since $m \geqslant 11$, the arcs of S are vertex disjoint. We have already noted that the arcs of S are not arcs of any cycle of τ_m. Thus, we can form the source extension of A_m with respect to S. The result is $AC_{m+1} = AC_{2n}$ and, by Lemma 16, we have $r(AC_{2n}) = 0$ since $r(A_{m-1}) = 0$. $\square$

6.6. *Arborescences*

An *arborescence* is a rooted tree on n vertices ($n \geqslant 2$), with the arcs directed so that there is a (directed) path from the root to every vertex. Let RT_n denote an arborescence on n vertices. It is well known that an arborescence on n vertices contains $n - 1$ arcs.

Theorem 24. *Let RT_n be an arborescence on n vertices. Then $r(RT_2) = 1$ and $2 \leqslant r(RT_n) \leqslant n - 1$ for $n \geqslant 3$ and the bounds are reached.*

Proof. The upper bound is immediate from Eq. (1) and the fact that arborescences on n vertices have $n - 1$ arcs. By Theorems 13 and 19, the upper bounds are attained by the directed paths P_n and stars with a unique source, both of which are arborescences.

For $n = 2$, the only arborescence on two vertices is the path P_2 with reversing number one (by Theorem 13). For $n = 3$, the only arborescences on three vertices are the alternating path A_3 and the directed path P_2, both with reversing number two (by Theorems 13 and 22).

Consider the case $n = 4$. Let RT_4 be any arborescence on four vertices. RT_4 has three arcs. Recall the results of Bermond and Kodratoff [6] regarding m_k, the largest number of arcs in a reversing set on a tournament on k vertices mentioned before the proof of Theorem 17. We have $m_k < 3$ for $k < 5$ so RT_4 is not a minimum reversing set of any tournament on four vertices, i.e., $r(RT_4) \geqslant 1$. It is easy to show (see [4]) that the only tournament on five vertices with a minimum reversing set of size $m_5 = 3$ is the regular tournament on five vertices (see Fig. 3), and that the minimum reversing sets of this tournament are not arborescences. So $r(RT_4) \geqslant 2$.

Fig. 18 gives an example of an arborescence on 4 vertices and a tournament on six vertices realizing it. This shows that the lower bound is attained for $n = 4$.

Finally, we consider $n \geqslant 5$. We must show the lower bound and show that this bound is attained.

Let ρ_n denote the minimum value of the reversing number for an arborescence on n vertices. We first show that $\rho_{n-1} \leqslant \rho_n$. Let RT_n be an arborescence such that $r(RT_n) = \rho_n$ and let T be a tournament on $n + \rho_n$ vertices realizing it. By Lemma 2, the vertex which is the unique source of $(T \setminus RT_n) \cup RT_n^R$ is a vertex of RT_n and thus must be a leaf of RT_n. Call this leaf x. Let RT_{n-1} denote the subarborescence of RT_n induced by $V(RT_n) \setminus \{x\}$ (i.e., the arborescence obtained by deleting x and its incident arc from RT_n). Also, let T' be the subtournament of T induced by $V(T) \setminus \{x\}$. By Lemma 3, RT_{n-1} is a minimum reversing set of T'. Thus $r(RT_{n-1}) \leqslant \rho_n$ (since

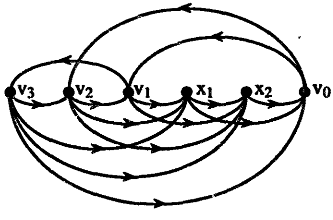


Fig. 18. A tournament on six vertices with an arborescence on four vertices as a minimum reversing set, containing arc disjoint cycles (v_3, v_2, v_1), (v_2, x_0, v_0) and (v_1, x_1, v_0).

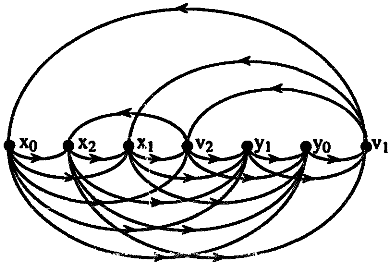


Fig. 19. A tournament on seven vertices with an arborescence on five vertices as a minimum reversing set, containing arc disjoint cycles (x_0, x_2, v_1), (x_2, x_1, v_2), (x_1, y_0, v_1), and (v_2, y_1, v_1).

$|V(T)\setminus V(RT_n)| = |V(T')\setminus V(RT_{n-1})| = \rho_n$). We have already shown $\rho_4 = 2$, so $\rho_n \geqslant 2$ for $n \geqslant 5$.

To show that the lower bound is attained, we exhibit first in Fig. 19 a tournament on seven vertices realizing an arborescence on five vertices.

For $n \geqslant 5$, we prove by induction that there exist tournaments T_n, arborescences RT_n (on n vertices), and collections τ_n of arc disjoint cycles satisfying:

(a) $|V(T_n)| = n + 2$.

(b) T_n has RT_n as a minimum reversing set.

(c) $|\tau_n| = n - 1$.

(d) RT_n has at least three leaves x_0, x_1, x_2.

(e) The arc (x_0, x_1) is in T_n but is not an arc of any cycle of τ_n.

Showing (a) and (b) will complete the proof as this shows that $r(RT_n) \leqslant 2$ and we already have that $r(RT_n) \geqslant 2$. So $r(RT_n) = 2$.

For $n = 5$, (a)–(e) are satisfied for the example of Fig. 19. The source extension of RT_n with respect to (x_0, x_1) gives RT_{n+1}, and the extensions of τ_n and T_n give τ_{n+1} and T_{n+1}. By construction of the extensions and by Lemma 16, (a)–(c) hold. Denoting the new vertex in the extension by v, we see that RT_{n+1} has leaves x_0, v and x_2. So (d) holds. Also, (e) holds for the arc (v, x_2) which is not an arc of any cycle of τ_{n+1}. □

7. Conclusion

The acyclic order obtained after reversal of the arcs in a minimum reversing set can be used as a ranking of the players in a round robin tournament. In this case the minimum reversing set represents inconsistencies in the ranking, those cases where player a beats player b but a is ranked below b. The reversing number is defined by the minimum number of additional vertices in a smallest tournament in which a given set of inconsistencies can arise. It would be interesting to determine the exact value within the bounds $2n - 4\log n \leqslant r(T_n) \leqslant 2n - 4$ of the reversing number of the acyclic tournament on n vertices. It would also be interesting to examine exact values of the reversing number on other classes of acyclic digraphs, or for example to find an expression for the exact value of the reversing number of any arborescence. Another open question is to determine bounds on $d(n, r)$, the largest arc size of a connected digraph on n vertices with reversing number r. We have not been able to show that $d(n, r + 1) > d(n, r)$, even though this seems plausible.

Calculation of the reversing number in general seems difficult. (Note that determining the reversing number would seem to require calculations of the size of minimum reversing sets and that that problem is NP-hard.) We currently do not know the complexity status of determining the reversing number. In fact, we do not even have algorithms for determining the reversing number for any class of acyclic digraphs.

Finally, recall that the minimum reversing sets arise as the sets of backwards arcs relative to a ranking which minimizes the number of backwards arcs. It would be possible to examine sets of arcs which arise as the backwards arcs under different ranking procedures, for example a ranking based on outdegrees. A similar question of determining the size of a smallest tournament in which a given acyclic digraph is the set of backwards arcs under an "optimal" ranking can be asked. (See [17], where this question is asked for a weighted version of the reversing number, which is equivalent to using a ranking based on score sequences.) Such computations might provide another way to evaluate ranking procedures for tournaments.

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