A Characterization of Graphs which Assure the Existence of Stable Matchings

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Abstract

In this note we characterize sets of mutually acceptable pairs of individuals in the stable matching problem for which a stable matching exists under all preference profiles. In particular, our characterization asserts that the corresponding acceptability graph is bipartite.

Key words: Stable matching; stable marriage; graph theory.

Abbreviated title: Stable matching.

1 Introduction

The stable marriage problem was introduced by Gale and Shapley (1962). In the original problem, there are n men and n women where each has a strict preference ordering of all the individuals of the opposite sex. A *stable marriage* is a one-to-one matching of the men with the women such that there is no man-woman pair that prefer each other over their present mates. Gale and Shapley describe a simple algorithm which, given any arbitrary profile of preference orderings, will find a stable marriage. This algorithm proves that stable marriages always exist.

The stable marriage problem can be extended to allow that the sets of men and women be of unequal size and cover the possibility of incomplete preference orderings (see e.g. Roth and Sotomayor, 1990). In this case, each person has a strict preference

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ordering over a subset of the individuals of the opposite sex, implicitly declaring that individuals outside that subset are unacceptable partners. If a man and a woman are on each other's list of acceptable partners, we say that they are *mutually acceptable*. A *matching* is a one to one pairing of mutually acceptable men and women. A man and a woman who are mutually acceptable will be called a *blocking pair* with respect to a given matching if the man is either unmatched or prefers the woman to his present mate and the woman is either unmatched or prefers the man to her present mate. A *stable marriage* is defined as a matching admitting no blocking pairs. A standard adjustment of the Gale-Shapley algorithm is guaranteed to find a stable matching even in the more general case where not all partners are acceptable.

Gale and Shapley also considered a generalization of the stable marriage problem called the *stable roommates problem*. Here we describe the variant which allows for singlehood. Specifically, there is a set of n individuals with each person having a preference ordering over a subset of the remaining people. Acceptability is defined as in the stable marriage problem and the individuals are to be matched in pairs which are mutually acceptable. Blocking pairs are defined in a manner analogous to those blocking stable marriages. The stable roommates problem is then to find a matching admitting no blocking pairs.

In contrast with the stable marriage problem, Gale and Shapley showed with the following simple example that there are situations in which no stable matching exists for the stable roommates problem.

Example 1:
Consider four people a, b, c, d with the following preference rankings:

$$\begin{aligned} P(a) = &\quad b, c, d \\ P(b) = &\quad c, a, d \\ P(c) = &\quad a, b, d \\ P(d) = &\quad \text{any ordering of } \{a, b, c\}. \end{aligned}$$

Here, for example, a prefers b to c to d. In this case, any incomplete matching is unstable since all pairs are mutually acceptable. In any complete matching, the person that is matched with d prefers being with any of the other two, and one of these ranks this person first. These two form a blocking pair. Thus there can be no stable matching in this example.

It is shown in Alkan (1988) that the generalization to the case when groups of three are matched also may not have a stable solution. We find it convenient to refer to the stable roommates problem as the *stable matching problem*. The stable marriage problem is then a special case of the stable matching problem in which the individuals are partitioned into two (disjoint) sets called sexes, and those acceptable to any given person must be of the opposite sex.

Irving (1985) and Gusfield and Irving (1989) give polynomial algorithms which, for the general stable matching problem, either find a stable matching or show that none exists. The purpose of this note is to characterize the acceptability relations which assure the existence of a stable matching.

We conclude the introduction by mentioning the books of Knuth (1976), Gusfield and Irving (1989) and Roth and Sotomayor (1990) for more details and references on stable matching and stable marriage. In particular, Roth and Sotomayor (1990) examine the stable marriage problem from a game theoretic viewpoint and give examples where this model is applied.

2 Acceptability Graphs

It is convenient to view the stable matching problem from a graph theoretic perspective. Recall that an (undirected) *graph* G is a pair (V, E) where V is a finite set called the *vertex set* of G and E is a subset of $\{\{v, w\} : v, w \in V\}$ and is called the *edge set* of G. For $v \in V$, we will denote by $N(v)$ the set of neighbors of v, i.e., the set of vertices w such that $\{v, w\} \in E$. A stable matching problem is then represented by a pair (G, P) where G is a graph and P is a mapping on the vertex set of G into strict ordering preferences over subsets of V, where $v \in V$ is mapped into a strict ordering preference over $N(v)$. If (G, P) represents a stable matching problem, G is called the *acceptability graph* and P the *profile*. For v in the vertex set of G, the set $N(v)$ represents the sets of individuals w such that v and w are mutually acceptable.

We can think of the vertex set of the acceptability graph as the set of individuals with the edges representing mutually acceptable pairs. Thus, the acceptability graph can be viewed as a social structure representing mutual acceptability. A *matching* is then represented by a list μ of disjoint edges from the graph G. A blocking pair v, w with respect to a matching is represented by an edge $\{v, w\} \in E$, such that either v is not contained in any edge of μ, or, $\{v, x\} \in \mu$ for some x, such that w is ranked higher than x in $P(v)$, with a similar condition holding for w. A matching represented by μ is stable if it admits no such pair v, w.

A graph is *bipartite* if its vertex set can be partitioned into two sets such that there are no edges joining two vertices in the same set. For the stable marriage problem, the structure of the acceptability graph is bipartite. The two disjoint sets of the acceptability graph of the stable marriage problem are the sets of men and women respectively. For the original stable roommates problem introduced by Gale and Shapley, the acceptability graph is a *complete graph*, all possible pairs are edges.

We have seen that there are instances where no stable matching exists. We next characterize the situations where, knowing only the structure of the acceptability graph, we can guarantee the existence of a stable matching under all profiles.

Theorem *Let G be a graph. Then (G, P) has a stable matching under all profiles P if and only if G is bipartite.*

Proof: The Gale-Shapley algorithm guarantees existence of a stable matching when G is bipartite.

To prove necessity, we show that if G is non-bipartite, then there is a profile for which there is no stable matching in G. A cycle in a graph is a set $v_1, v_2, \ldots, v_k$ of distinct vertices such that $\{v_i, v_{i+1}\} \in E$ is an edge and $\{v_k, v_1\}$ is an edge. It is well known that if a graph is non-bipartite, then it must contain an odd cycle. Let $v_1, \ldots, v_{2p+1}$ be an odd cycle in G. We consider the following profile, where each vertex on the odd cycle ranks its neighbors first and second. Let $v_0 \equiv v_{2p+1}$ and $v_{2p+2} \equiv v_1$.

$$P(v_i) = v_{i+1}, v_{i-1}, \text{ any ordering of } N(v_i) \setminus \{v_{i-1}, v_{i+1}\} \text{ if } i = 1, \ldots, 2p+1$$

and

$$P(v) = \text{ any ordering of } N(v) \text{ if } v \notin \{v_1, \ldots, v_{2p+1}\}.$$

Consider a matching μ. There must be at least one vertex among $\{v_1, \ldots, v_{2p+1}\}$ which will not be matched to another vertex on the cycle. Without loss of generality, we can assume that v_2 is such a vertex. Then v_2 prefers v_1 to its current situation (whether or not v_2 is matched under μ). Also, v_2 is v_1's first choice. So, v_1 prefers v_2 to its current situation. Thus v_2 and v_1 form a blocking pair and μ is not stable. □

The main purpose of this note was to introduce the notion of acceptability graphs in the study of the stable matching problem and examine conditions for stability based on the structure of this graph. The acceptability graphs can be viewed as a representation of the underlying social structure of mutually acceptable pairs. We have shown that the social structures which guarantee stability under all profiles are precisely those where the acceptability graph is bipartite. A forthcoming paper by Abeledo (1990) studies conditions which are necessary and sufficient for the existence of stable matchings in non-bipartite graphs.

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