Hamiltonian Powers in Threshold and Arborescent Comparability Graphs

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Abstract

We examine powers of Hamiltonian paths and cycles as well as Hamiltonian (power) completion problems in several highly structured graph classes. For threshold graphs we give efficient algorithms as well as sufficient and minimax toughness like conditions. For arborescent comparability graphs we have similar results but also show that for one type of completion problem an 'obvious' minimax condition fails. For cographs we give examples showing that toughness and other 'obvious' necessary conditions are not sufficient. For threshold graphs we give additional necessary and sufficient conditions in terms of vertex degrees as well as a minimax formula for the length of a longest cycle power.

1 Introduction

A graph G contains the k^{th} power of a Hamiltonian path if the vertices can be ordered so that vertices at distance k or less are adjacent in G. A graph contains the k^{th} power of a Hamiltonian cycle if the ordering is cyclic. That is, G contains the k^{th} power of a Hamiltonian path if the vertices can be labeled $1, 2, \ldots, |V|$ such that $|i - j| \leq k$ implies $ij \in E$ and the k^{th} power of a Hamiltonian cycle if the vertices can be labeled $1, 2, \ldots, |V|$ such that $|i-j| \leq k$ or $|i-j| \geq |V|-k$ implies $ij \in E$. If $|V| \leq k$ we will say that the graph contains a k^{th} power of a Hamiltonian path and cycle if and only if it is complete. The cases $k = 1$ are paths and cycles (except our definition includes a single vertex or edge as a cycle). A k path power in a graph is an induced subgraph which contains the k^{th} power of a Hamiltonian path and similarly for k cycle powers.

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Analogous to Hamiltonian completion problems we look at vertex completion, the minimum number of additional 'new' vertices adjacent to all vertices so that G contains the k^{th} power of a Hamiltonian path and path partition, the minimum number of k path powers needed to cover the vertices.

Toughness was introduced by Chvátal [2] as a basic necessary (but not sufficient) condition for the existence of Hamiltonian cycles in a graph. See the definition below. It is conjectured that 2-toughness is sufficient for the existence of a Hamiltonian cycle in general graphs. Not surprisingly an analogous conjecture for k^{th} powers of Hamiltonian cycles fails. In [1] triangle free graphs with arbitrarily large toughness are constructed. Such graphs can not contain even a square of a Hamiltonian cycle. For highly structured graph classes, such a cocomparability graphs (and hence cographs, interval graphs, threshold graphs etc), 1-toughness is sufficient for the existence of Hamiltonian cycles [5], [6], [7].

Our aim in this paper is to look at graph classes where k-toughness is sufficient for the existence of a k^{th} power of a Hamiltonian cycle and also to look where this fails. We will also examine minimax results for related Hamiltonian (power) completion problems.

The following gives definitions and easily verified facts relating toughness like conditions and Hamiltonian powers. (See [12] for more details.) Let $C(S)$ denote the number of components in the graph induced by $V - S$.

- If G contains the k^{th} power of a Hamiltonian path then

$$|S| \geq k(C(S) - 1) \text{ for all } S \subseteq V. \tag{1}$$

  We call a graph satisfying the condition in (1) k-path tough. So k-path toughness is a necessary condition for a graph to contain the k^{th} power of a Hamiltonian path.

- If G contains the k^{th} power of a Hamiltonian cycle then

$$|S| \geq kC(S) \text{ or } C(S) = 1 \text{ for all } S \subseteq V. \tag{2}$$

  A graph satisfying the condition in (2) is k-tough. So k-toughness is a necessary condition for a graph to contain the k^{th} power of a Hamiltonian cycle.

- The vertex completion number of G, denoted $VC_k(G)$, is the minimum t such that the join of G and K_t, denoted $G \vee K_t$, contains the k^{th} power of a Hamiltonian path.

$$VC_k(G) \geq \max_{S \subseteq V} \left\{ k(C(S) - 1) - |S| \right\}. \tag{3}$$

- The path partition number of G, denoted $PP_k(G)$, is the minimum t such that there exist $Q_1, Q_2, \ldots, Q_t$ partitioning V and for each i, Q_i is a k path power.

$$PP_k(G) \geq \max_{S \subseteq V} \left\{ C(S) - \left\lfloor \frac{|S|}{k} \right\rfloor \right\}. \tag{4}$$

In general the necessary conditions above are not sufficient and the bounds are not tight. Our results and examples examine when the conditions are sufficient and when equality holds in the bounds. We can summarize the results as follows.

- For arborescent comparability graphs (and hence threshold graphs) the necessary toughness conditions for Hamiltonian path and cycle powers are also sufficient. We can give efficient algorithms to test for such powers on these graph classes.

- For cographs the necessary toughness conditions for Hamiltonian path and cycle powers are not sufficient. We give infinite families of cographs that satisfy the toughness conditions but do not contain the corresponding Hamiltonian path or cycle powers.

- For arborescent comparability graphs (and hence threshold graphs) equality holds in (3) for vertex completion. That is,

$$VC_k(G) = \max_{S \subseteq V} \left\{ k(C(S) - 1) - |S| \right\}.$$

  For cographs, the inequality can be strict and the gap can be arbitrarily large.

- For threshold graphs equality holds in (4) for path partitions. That is,

$$PP_k(G) = \max_{S \subseteq V} \left\{ C(S) - \left\lfloor \frac{|S|}{k} \right\rfloor \right\}.$$

  Additionally, for threshold graphs a partition can be found that consists of one 'big' part Q_1 with the remaining parts consisting of isolated vertices. For arborescent comparability graphs (and hence cographs) the inequality can be strict and the gap can be arbitrarily large.

Actually, the sufficiency of the toughness conditions follows from vertex completion results but we have stated them separately here to emphasize the different problems. Some of the path results follow from [12].

For threshold graphs, we provide additional results, giving necessary and sufficient conditions for Hamiltonian powers in terms of vertex degrees, a minimax theorem on the length of a longest cycle power and power 'pancyclic' conditions.

2 Cographs, Arborescent Comparability and Threshold Graphs

In this section we briefly review definitions and representations of the various graph classes of interest here. For more information and history see for example [4], [10], [11] or [15]. For more general graph theory terms see [16].

The join of two graphs on disjoint vertex sets, denoted $G \vee H$ is formed by taking a copy of G and a copy of H and adding all possible edges between the two. The union $G \cup H$ consists of disjoint copies of G and H ($G \vee H$ without the edges between G and H.) The disjoint union of m copies of G will be denoted mG. We will use $\overline{G}$ to denote the complement of G, P_n and C_n to denote 'the' path and 'the' cycle on n vertices and K_n to denote 'the' complete graph on n vertices. The open neighborhood $N(v)$ of a vertex is the set of adjacent vertices and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. A comparability graph of an order has vertex set corresponding to the elements of the order and xy an edge if and only if the corresponding elements are related in the order.

Cographs: Graph G is a cograph if and only if it does not contain an induced P_4 (path on 4 vertices). Alternatively, G is a cograph if and only if it can be expressed in terms of complete graphs using $\cup$ (union), $\vee$ (join) and complementation (actually any two of these three operations suffices). Cographs are comparability graphs (and also cocomparability graphs) of series parallel orders.

Arborescent Comparability Graphs: Graph G is an arborescent comparability graph if and only if it does not contain an induced P_4 or an induced C_4. Alternatively (and the origin of the name), G is an arborescent comparability graph if it is the comparability graph of an order P whose diagram is a set of rooted forests, i.e., for $x \in P$, $\{y | y \succ x\}$ is a chain. We will call the set of elements (vertices) preceding all other elements (adjacent to all other vertices) the root chain. There is a unique (up to isomorphism) order representing a given arborescent comparability graph. We can also express an arborescent comparability graph in terms of complete graphs, using $\cup$ and $\vee$ with the condition that $\vee$ can only be used as $K_n \vee H$ where H is an arborescent comparability graph.

Threshold Graphs: Graph G is a threshold graph if and only if it does not contain an induced P_4 or an induced C_4 or an induced $K_2 \cup K_2 = 2K_2$. Alternatively, we can partition the vertices into (a possibly empty) set of isolated vertices D_0 and non empty sets $D_1, D_2 \ldots, D_m$ such that $x \in D_i$ and $y \in D_j$ are adjacent if and only if $i + j > m$. This is called the degree partition. Also note that $D_0 \cup D_1 \cup \cdots D_{\lfloor m/2 \rfloor}$ is an independent set and $D_{\lceil m/2 \rceil} \cup \cdots D_{m-1} \cup D_m$ is a clique. We can also express a threshold graph in terms of complete graphs, using $\cup$ and $\vee$ with the conditions that $\cup$ can only be used as $K_n \cup H$ where H is a threshold graph and $\vee$ can only be used as

$K_n \vee H$ where H is a threshold graph. Let $p(i) = \sum_{j=1}^{i} |D_{m+1-i}|$. A threshold graph is the comparability graph of an order which consists of $|D_0|$ isolated elements along with a chain $x_1, x_2, \ldots, x_{p(1)}, \ldots, x_{p(\lceil m/2 \rceil)}$ such that for $i = 1, 2, \ldots, \lfloor m/2 \rfloor$ there are $|D_i|$ leaves attached to $x_{p(i)}$.

Observe that every threshold graph is an arborescent comparability graph and every arborescent comparability graph is both an interval graph (see [11]) and a cograph.

3 Cycles, Paths and Vertex Completion

In this section we show a minimax result for vertex completion in arborescent comparability graphs (and hence in threshold graphs). This yields an efficient algorithm as well as sufficiency of the toughness conditions. Note that the cycle results imply the path results (see similar observations in [13], [12]), but for the inductive proof it is easier to do it all together. Also note that the results for paths and vertex completion (but not cycles and not the structure of the sets S) are implied by results for interval graphs in [12].

In order to simplify the proof we need one new definition. The vertex cycle completion number, denoted $VCC_k(G)$, is the minimum t such that $G \vee K_t$ contains the k^{th} power of a Hamiltonian cycle. Observe that two immediate corollaries will be sufficiency of the toughness conditions for paths and cycles.

Theorem 1 *Let G be an arborescent comparability graph. Then*

$$VC_k(G) = \max_{S \subseteq V} \{k(C(S) - 1) - |S|\}$$

and

$$VCC_k(G) = \max_{S \subseteq V, C(S) > 1} \{kC(S) - |S|, 0\}.$$

Furthermore, when the maximum is at least 1, there is an S attaining the maximum which corresponds to an upper order ideal in the arborescent order representing G.

Proof: In each case $\geq$ is simply an extension of the necessity of the toughness conditions and easy to check.

To show $=$ we will use induction on the number of vertices. If G is complete the results are obvious.

Let R with $|R| = r$ (possibly empty) be the root chain. That is, $G = K_r \vee (H_1 \cup H_2 \cup \cdots \cup H_c)$. If R is empty then $c \geq 2$ so we can assume the result holds for the components H_i. If the maximum is 0, let $S_i = \emptyset$. Otherwise, let S_i be an upper order ideal attaining the maximum in the H_i.

It is easy to see that if

$$\rho = \sum_{i=1}^{c} VC_k(H_i) + k(c-1)$$

then $K_\rho \vee (\cup_{i=1}^{c} H_i)$ has a k^{th} power of a Hamiltonian path. (We use $\sum_{i=1}^{c} VC_k(H_i)$ vertices to finish path powers in the H_i and $k(c-1)$ vertices to 'patch' together the pieces H_i.) Similarly if

$$\sigma = \sum_{i=1}^{c} VC_k(H_i) + kc$$

then $K_\sigma \vee (\cup_{i=1}^{c} H_i)$ has a k^{th} power of a Hamiltonian cycle. Note that the pieces that are 'patched' together are Hamiltonian paths, not cycles, so we will use $VC_k(H_i)$ inductively, not $VCC_k(H_i)$.

We show the result for paths first. If $|R| = r \geq \rho$ then G contains the k^{th} power of a Hamiltonian path and we are done. If not, the comments of the previous paragraph show that $VC_k(G) \leq \rho - r$. So we need to show that there exists an upper order ideal S with $k(C(S)-1) - |S| = \rho - r$. Observe this will also show that the case when the maximum is 0 implies $|R| \geq \rho$.

Let $C_i(S_i)$ denote the number of components when S_i is deleted from H_i. So $VC_k(H_i) = k(C_i(S_i) - 1) - |S_i|$ for $i = 1, 2, \ldots, c$.

Let $S = R \cup (\cup_{i=1}^{c} S_i)$ and observe that $|S| = r + \sum_{i=1}^{c} |S_i|$ and $C(S) = \sum_{i=1}^{c} C_i(S_i)$. Then

$$\begin{aligned} k(C(S)-1) - |S| &= k\left(\left[\sum_{i=1}^{c} C_i(S_i)\right] - 1\right) - r - \sum_{i=1}^{c} |S_i| \\ &= \sum_{i=1}^{c} [kC_i(S_i) - |S_i|] - k - r \\ &= \sum_{i=1}^{c} [k(C_i(S_i) - 1) - |S_i|] + (c-1)k - r \\ &= \rho - r \end{aligned}$$

The proof for cycle powers is the same except add k to each term in the above sequence of equations. $\square$

The following corollary is immediate, we state it for completeness.

Corollary 1 *For an arborescent comparability graph G, k-path toughness is a necessary and sufficient condition for the k^{th} power of a Hamiltonian path and k toughness is a necessary and sufficient condition for the k^{th} power of a Hamiltonian cycle. Furthermore, if G is not k-path tough, there is a set S corresponding to an upper order ideal with $|S| < k(C(S)-1)$ and if G is not k-tough there is a set S corresponding to an upper order ideal with $C(S) > 1$ and $|S| < kC(S)$.*

Since threshold graphs are also arborescent comparability graphs the sufficiency in the next corollary is the same as the previous, however this corollary specifies the form of 'violating' sets in threshold graphs.

Corollary 2 *For a threshold graph G k-path toughness is a necessary and sufficient condition for the k^{th} power of a Hamiltonian path and k toughness is a necessary and sufficient condition for the k^{th} power of a Hamiltonian cycle. Furthermore, if G is not k-path tough, there is a set S of the form $D_m \cup D_{m-1} \cup D_{m-j}$ for some $j < \lfloor m/2 \rfloor$ with $|S| < k(C(S) - 1)$ and if G is not k-tough there is a set S of the form $D_m \cup D_{m-1} \cup D_{m-j}$ for some $j < \lfloor m/2 \rfloor$ with $C(S) > 1$ and $|S| < kC(S)$.*

Proof: This follows from the description of orders representing threshold graphs and from observing that if a set is an upper order ideal violating the condition for toughness and it contains a leaf then removing that leaf from the set produces a new set which also violates the condition. □

It is easy to see how the inductive proof of Theorem 2 can be translated into a recursive algorithm. So we have the following corollary. We will not go into details of implementation.

Corollary 3 *There are efficient algorithms for Hamiltonian cycle power, Hamiltonian path power and vertex completion in arborescent comparability graphs.*

For threshold graphs, additional very natural algorithms can be found in [8].

4 Threshold Graphs

In this section we will discuss various additional structural results that can be obtained for threshold graphs, including path partitions, longest cycle length, power pancyclicity and various conditions in terms of degrees and the degree partition.

Throughout this section we will assume that the vertices of a threshold graph are labeled with non-decreasing degrees. So $(v_1, v_2, \ldots, v_{|D_0|} \in D_0)$, $(v_{|D_0|+1}, \ldots, v_{|D_0|+|D_1|} \in D_1)$, …, $(v_{1+\sum_{i=1}^{m-1}|D_i|}, \ldots, v_{\sum_{i=1}^{m}|D_i|} \in D_m)$.

We will show in the next section that the bound for path power partitions noted in the introduction is not tight for arborescent comparability graphs. Here we show that it is tight for threshold graphs.

Theorem 2 *If G is a threshold graph then*

$$PP_k(G) = \max_{S \subseteq V} \left\{ C(S) - \left\lfloor \frac{|S|}{k} \right\rfloor \right\}.$$

Additionally, a partition can be found that consists of one 'big' part Q_1 with the remaining parts consisting of isolated vertices.

Proof: Let $\gamma = \max_{S\subseteq V}\left\{C(S) - \left\lfloor \frac{|S|}{k} \right\rfloor\right\}$ and let $D_0, D_1, \ldots, D_m$ be the degree partition of G.

Consider first $\gamma = |V|$. In this case, the graph consists of isolated vertices, so the size of a minimum path partition is $|V|$. Since $|V| = \gamma = \max_{S\subseteq V}\left\{C(S) - \left\lfloor \frac{|S|}{k} \right\rfloor\right\}$ (with $S = \emptyset$), we are done.

Next consider $\gamma = 1$. Then for all $S \subseteq V$, $C(S) - \left\lfloor \frac{|S|}{k} \right\rfloor - 1 \leq 0$. So $C(S) - 1 \leq \left\lfloor \frac{|S|}{k} \right\rfloor \leq \frac{|S|}{k}$. Then $k\left(C(S) - 1\right) \leq |S|$ for all $S \subseteq V$, i.e., G is k-path tough. So, by Corollary 2, G contains a k^{th} power of a Hamiltonian path and $PP_k(G) = 1$ as required.

So assume that $1 < \gamma < |V|$. Let $R = \{v_1, v_2, \ldots, v_{\gamma-1}\}$ (recall the label of vertices by non-decreasing degree) and let G' be the subgraph induced by $V - R$. We will show that G' contains the k^{th} power of a Hamiltonian path. If $\gamma - 1 \geq \sum_{i=1}^{\lfloor m/2 \rfloor} |D_i|$ then G' is a complete graph and so must contain the k^{th} power of a Hamiltonian path. So we may also assume that $\gamma - 1 < \sum_{i=1}^{\lfloor m/2 \rfloor} |D_i|$.

Assume towards a contradiction that G' does not contain a k^{th} power of a Hamiltonian path. G' is an induced subgraph of a threshold graph and so is also a threshold graph. By Corollary 2, there exists S' such that $|S'| < kC_{G'}(S') - k$.

We claim that $C_G(S) = C_{G'}(S') + |R|$. Since $\gamma - 1 < \sum_{i=1}^{\lfloor m/2 \rfloor} |D_i|$, R is an independent set. So, if $C_G(S) \neq C_{G'}(S') + |R|$, there is some $v \in R$ and some $u \in V(G') - S'$ such that $vu \in E(G)$. Then, if $v \in D_i$ and $u \in D_j$, we have $i + j > m$ from the definition of degree partition in a threshold graph. It follows because of the way that we formed R, that for every $v' \in V - R$, $v' \in D_{i'}$ for $i' \geq i$ and thus $i' + j > m$ and $v'u \in E$. So $C_{G'}(S') = 1$ and $|S'| < kC_{G'}(S') - k = 0$, which is impossible. So $C_G(S) = C_{G'}(S') + |R|$.

Since $|S'| < kC_{G'}(S') - k$ we have $C_{G'}(S') - 1 - \left\lfloor \frac{|S'|}{k} \right\rfloor > 0$. Now,

$$\begin{aligned}
\gamma &\geq C_G(S') - \left\lfloor \frac{|S'|}{k} \right\rfloor \\
&= C_{G'}(S') + |R| - \left\lfloor \frac{|S'|}{k} \right\rfloor \\
&= C_{G'}(S') + \gamma - 1 - \left\lfloor \frac{|S'|}{k} \right\rfloor \\
&> \gamma
\end{aligned}$$

a contradiction. So G' contains the k^{th} power of a Hamiltonian path and $Q_1, Q_2, \ldots, Q_\gamma$ with $Q_1 = V - R$ and $Q_i = v_{i-1}$ for $i = 2, \ldots, \gamma$ is a path power partition with all parts except Q_1 consisting of a single vertex. $\square$

The partition contains one long path power and isolated vertices. It is not hard to see that the path power in the 'big' part is also a longest path power in a threshold graph. This follows by observing the 'nested' property of the neighborhoods. If P is a longest path power in a threshold graph G and $v_i \in P$, $v_j \notin P$ with $i \leq j$, i.e., the degree of v_j is at least the degree of v_i, then replacing v_i in P with v_j yields a path power of the same length. This follows since in this case $xv_i \in E$ implies $xv_j \in E$. Hence the maximum size of a longest path power in a threshold graph is the size of the 'big' part,

$$\begin{aligned}
\text{maximum size of a path } k\text{-power} &= |V| - (PP_k(G) - 1) \\
&= |V| + 1 - \max_{S \subseteq V} \left\{ C(S) - \left\lfloor \frac{|S|}{k} \right\rfloor \right\} \\
&= \min_{S \subseteq V} \left\{ |V| + 1 + \left\lfloor \frac{|S|}{k} \right\rfloor - C(S) \right\}.
\end{aligned}$$

In a similar manner we can look for the longest cycle power by deleting vertices in an order from lowest degree to highest until the remaining graph contains a cycle power. The longest cycle in a threshold graph is discussed in [14], [15]. Our perspective here is slightly different. We seek another sort of minimax type formula. The proof of the following theorem is omitted since it is nearly identical to the proof of Theorem 2 and the remarks in the paragraph following that proof.

Theorem 3 *Let G be a threshold graph which does not contain the k^{th} power of a Hamiltonian cycle, then*

$$\textit{maximum size of a cycle } k \textit{ power } = \min_{S \subseteq V} \left\{ |V| + \left\lfloor \frac{|S|}{k} \right\rfloor - C(S) \right\}.$$

Observe that except when the graph contains a k^{th} power of a Hamiltonian path the maximum size of a k path power is exactly one more than the maximum size of a k cycle power.

Theorem 1.6.10 in [15] states that if L is the length of a longest cycle in a threshold graph G (or even an arborescent comparability graph) then G contains cycles of lengths $3, \ldots, L$. So a Hamiltonian threshold graph is pancyclic. Their (short) proof uses forbidden induced C_4 and P_4. We give an alternative proof, which is also elementary, that applies to cycle powers in the larger class of triangulated graphs. Recall that when $k = 1$ our definition includes a single vertex or edge as a cycle. This is probably well known for the case $k = 1$.

Remark 1 *If G is a triangulated graph with longest k cycle power of size L, then G contains k cycle powers with sizes* $1, 2, \ldots, L$

Proof: Observe that if a cycle power contains a vertex whose open neighborhood is a clique then deleting that vertex from the cycle power leaves a new cycle power with one less vertex. It is well known that triangulated graphs have such a simplicial vertex. The graph induced by a longest cycle power in a triangulated graph is triangulated and hence has a simplicial vertex. Delete such a vertex and repeat with the shorter cycle power. □

Note that this says that triangulated graphs with the k^{th} power of a Hamiltonian cycle are k cycle power 'pancyclic'.

Conditions for Hamiltonian cycles in threshold graphs in terms of degree sequences and degree partitions are reported in [3] and [11] among others (see also [15]). Here we give the k power analogues.

By noting from Corollary 2 the special form of sets violating the condition of (2) and translating this into degree partitions and degrees we easily get the following (proofs are omitted).

Corollary 4 *If G is a threshold graph with degree partition $D_0, D_1, \ldots, D_m$, then G contains the k^{th} power of a Hamiltonian cycle if and only if*

1. $D_O = \emptyset$;
2. $k\left(1 + \sum_{i=1}^{j} |D_i|\right) \leq \sum_{i=1}^{j} |D_{m-i+1}|$ *for* $j = 1, 2, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor$;
3. *if m is even, then* $k \sum_{i=1}^{m/2} |D_i| \leq \sum_{i=1}^{m/2} |D_{m-i+1}|$.

Corollary 5 *If $d_1 \leq d_2 \leq \cdots \leq d_n$ are the degrees of a threshold graph G then G contains the k^{th} power of a Hamiltonian cycle if and only if there is no $j < \frac{n}{k+1}$ with $d_j < (j+1)k$.*

Results such as Theorems 2 and 3 could also be easily translated into the language of degree partitions or degrees using the structure result of Corollary 2 but we will not do so here.

5 Examples

In this section we give examples showing where the theorems of the previous sections can not be extended to larger families. We also examine some other (non toughness) conditions related to Hamiltonian powers.

We first observe another obvious necessary condition for Hamiltonian powers.

Lemma 1 *If G contains the k^{th} power of a Hamiltonian cycle then the open neighborhood $N(v)$ of every vertex v must contain at least two vertex disjoint $k-$cliques.*

Lemma 2 *If G contains the k^{th} power of a Hamiltonian path then at least $|V|-2$ vertices v must have open neighborhood $N(v)$ containing at least two vertex disjoint $k-$cliques.*

Next we give very simple examples of cographs where toughness conditions are not sufficient. Remark 4 will also provide such examples but those are more complicated.

Remark 2 *For all t there exist t-path tough cographs which do not contain the t^{th} power of a Hamiltonian Path and t-tough cographs which do not contain the t^{th} power of a Hamiltonian cycle.*

Proof: For arbitrary t and $m \geq 2t-1$, consider $(K_1 \cup K_{2t-1}) \vee (K_1 \cup K_m)$. This is a 4-cycle a, b, c, d with vertex b replaced by a $(2t-1)-$clique and vertex c replaced with an $m-$clique. By construction or by observing that the graph contains no induced P_4 we see that this is a cograph. It is easy to check that this graph is t-tough. By Lemma 1 applied to either of the K_1 vertices we see that the graph does not contain the t^{th} power of a Hamiltonian cycle.

Similar considerations show that for arbitrary t and $m \geq t-1$, the cograph $(K_1 \cup K_{t-1}) \vee (K_1 \cup K_m)$ is t-path tough but does not contain the t^{th} power of a Hamiltonian path. $\square$

Recall that for threshold graphs $PP_k(G) = \max_{S\subseteq V}\left\{C(S) - \left\lfloor\frac{|S|}{k}\right\rfloor\right\}$. This is not so for arborescent comparability graphs.

Remark 3 *There exist arborescent comparability graphs for which $PP_k(G) - \max_{S\subseteq V}\left\{C(S) - \left\lfloor\frac{|S|}{k}\right\rfloor\right\}$ is arbitrarily large.*

Proof: For arbitrary $r, s \geq 2$ consider the graph $G(r,s) = r(K_1 \vee sK_1)$, the disjoint union of r stars each with s leaves. This is the comparability graph of an order whose diagram is a forest of r stars, each with s leaves and hence an arborescent comparability graph. It is not difficult to check that

$$\max_{S\subseteq V}\left\{C(S) - \left\lfloor\frac{|S|}{k}\right\rfloor\right\} = rs - \left\lfloor\frac{r}{k}\right\rfloor$$

attained by picking the centers of the stars. Since two leaves can not be in the same part in a path partition (for $k \geq 2$), we can easily check that $PP_k(G(r,s)) = rs$ if $k \geq 2$. $\square$

The previous examples had vertices with low degree or vertices with only one 'large' clique in their neighborhood. This suggests adding conditions derived from Lemmas 1 and 2. Even for cographs this is not enough. The next example is motivated by an example from [9].

Remark 4 *There exist t-path tough cographs such that every vertex neighborhood contains at least two cliques of size at least t which do not contain the t^{th} power of a Hamiltonian path.*

Proof: Assume $r \geq s \geq 6$. Let P be an order with diagram consisting of r complete binary trees of height s. Let $H(r,s)$ be the cocomparability graph of this order. The comparability graph is P_4 and C_4 free, hence the cocomparability graph is P_4 and $2K_2$ free and $H(r,s)$ is a cograph. $H(r,s)$ has $n = r(2^s - 1)$ vertices.

Let $t = \lfloor n/s \rfloor$. We will show that $H(r,s)$ is t-path tough, every vertex contains at least two cliques of size at least t and $H(r,s)$ does not contain the t^{th} power of a Hamiltonian path.

We will refer to vertices in a subtree as those corresponding to the elements of a subtree in the order. Observe that if vertices from two different trees remain after removing a set S, then $G - S$ is connected (i.e., $C(S) = 1$). So we need only look at the case that S contains all vertices except some from one of the trees. That is, we must have $|S| \geq (r-1)(2^s - 1)$. Observe that two components of $G - S$ can not have vertices at the same level in the tree since H is a cocomparability graph. Thus $C(S) \leq s$. If $C(S) \leq s - 1$ it is easy to check that $|S| \geq t(C(S) - 1)$ (using $|S| \geq (r-1)(2^s - 1)$ and $r \geq s/2$). In order for $C(S) = s$, S must consist of all vertices except those of a chain from a root to a vertex covering leaves (in the order) and the two leaves it covers. That is, $|S| \geq n - (s+1)$. Using $r \geq s$ it is easy to check that $|S| \geq t(C(S) - 1)$ in this case.

In order to show that every vertex has a neighborhood containing two cliques of size at least t, note that the leaves of the trees form a clique. There are 2^{s-1} leaves in each tree. Each vertex has in its neighborhood the vertices of $r - 1$ other trees. So taking the leaves of $\lfloor (r-1)/2 \rfloor$ trees as one clique and the leaves of the remaining trees (not containing the vertex) as another clique we get two cliques with size at least $2^{s-1}(r-2)/2$. This is at least t if $s \geq 5$. (Since other vertices could be in the clique, with some more care we could reduce the $s \geq 5$ condition.)

Finally, to show that $H(r,s)$ does *not* contain the t^{th} power of a Hamiltonian path, assume that there is one, and the vertices are ordered $1, 2, 3, \ldots, n$ with vertices at distance t or less adjacent. Note that every vertex is in an independent set of size s, in particular the vertex labeled $t+1$. Pick some independent set of size s containing vertex $t+1$ and assume the labels are $x_1 < x_2 < \cdots < x_s$. Since $t+1$ is adjacent to $1, 2, \ldots, t$, we have $x_1 = t+1$. Similarly, $x_2 \geq 2(t+1)$, then $x_3 \geq 3(t+1)$, ... and $x_s \geq s(t+1)$. With $x_s \leq n$ and $t = \lfloor n/s \rfloor$ this is a contradiction. $\square$

6 Conclusion

We have examined various Hamiltonian power problems in certain structured graph classes. One variation that we have not looked at is a power analogue of edge completion. This seems to be tricky even for threshold graphs. Since toughness and the vertex neighborhood conditions are not sufficient conditions in cographs, it remains to examine this class, for efficient algorithms and necessary and sufficient conditions (or for NP-completeness). Perhaps something can even be said for larger classes, such as cocomparability graphs.

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References

[1] D. Bauer, J. van den Heuval and E. Schmeichel, Toughness and triangle free graphs, J. Comb. Theory B 65 (1995) 208–221.

[2] V. Chvátal, Tough Graphs and Hamiltonian Circuits, Disc. Math. 5 (1973) 215–228.

[3] V. Chvátal and P.L. Hammer, Aggregation of Inequalities in Integer Programming, Ann. Disc. Math. 1 (1977) 145–162.

[4] D. Corneil, H. Lerchs and L. Stewart Burlingham, Complement Reducible Graphs, Disc. Appl. Math. 3 (1981) 163–174.

[5] P. Damaschke, J.S. Deogun, D. Kratsch and G. Steiner, Finding Hamiltonian Paths in Cocomparability Graphs Using the Bump Number Algorithm, Order 8 (1992) 383–391.

[6] J.S. Deogun, D. Kratsch and G. Steiner, 1-tough Cocomparability Graphs are Hamiltonian, Disc. Math. 170 (1997) 99–106.

[7] J.S. Deogun, and G. Steiner, Polynomial Algorithms for Hamiltonian Cycle in Cocomparability Graphs, SIAM J. Comput. 23 (1994) 520–552.

[8] S. Donnelly, Powers of Hamiltonian Paths and Powers of Hamiltonian Cycles in Threshold Graphs, Honors Thesis, Dept. of Math., Lehigh University, Bethlehem, 1997.

[9] J.R. Driscoll, D.M. Healy and G. Isaak, Scheduling Dyadic Intervals, Disc. Appl. Math. 63 1995 101–116.

[10] P. Duchet, Classical Perfect Graphs, Ann. Disc. Math. 21 (1984) 67–96.

[11] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, 1980).

[12] G. Isaak, Powers of Hamiltonian Paths in Interval Graphs, J. Graph Theory 27 (1998) 31–38.

[13] D. Kratsch, J. Lehel, and H. Muller, Toughness, Hamiltonicity and Split Graphs, Disc. Math. 150 (1996) 231–245.

[14] N.V.R. Mahadev, and U.N. Peled, Longest Cycles in Threshold Graphs, Disc. Math. 135 (1994) 169–176.

[15] N.V.R Mahadev and U.N. Peled, Threshold Graphs and Related Topics, Ann. Disc. Math. 56 (1995).

[16] D.B. West, Introduction to Graph Theory (Prentice Hall, 1996).